



Waldorf Science

Explorations in Phenomenology as Practiced in Waldorf Schools

A Science News Roundletter

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VOL. 7, #13

Autumn 2000

NOTE FROM THE EDITORS

This edition of the newsletter has a focus on mathematics. We feature articles by Ernst Schuberth, Erik Marstrander, and Mick Follarri. We include an overview of the mathematics curriculum for Waldorf schools recently prepared in Europe under the supervision of Martyn Rawson. While we might do things somewhat different here in North America, the editors wanted to include it for your study and reference. Please see page #2 for information on how you can get a copy of Research Project #3 – the Proceedings of the Waldorf Mathematics Colloquium.

As always, we continue to ask that you please send us your favorite experiments, stories, ideas, and questions or gather them from your colleagues. Please let us know your needs and what you would like to see us concentrate on in future issues.

MATHEMATICS COLLOQUIUM

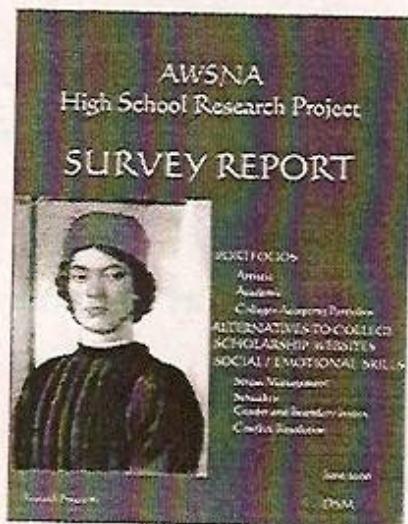
In October 2000, a colloquium was held to study the qualities of thinking that is represented in the various topics of mathematics in Waldorf high schools. This project was supported by the High School Research Group of AWSNA and included the following participants:

Cynthia Bennett, Remedial teacher, Bethesda, MD
David Booth, PhD., teacher, Austin Waldorf High School
Connie Gerwin, teacher, High Mowing School
Ronald Jarman, Stroud, Glos., England
Harry Kretz, teacher, Harlemville, NY
Martin Levin, PhD., teacher, Washington Waldorf High School
Erik Marstrander, teacher, Rudolfsteinerskole, Oslo, Norway
David Mitchell, MEd., AWSNA High School Research Group
Marisha Plotnik, MEd, Rudolf Steiner School, Manhattan
Bonnie River, MEd, Remedial teacher, Boulder, CO
Ernst Schuberth, PhD., Mannheim, Germany

BOOKS

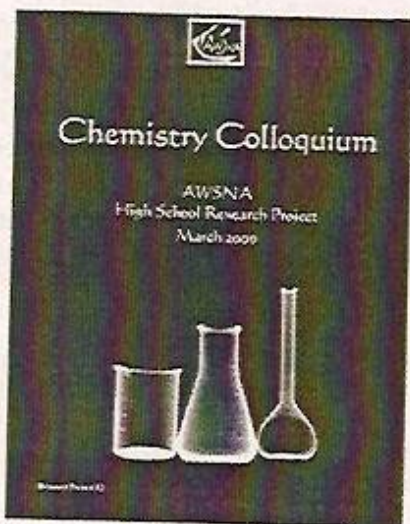
The following titles, produced by the Waldorf High School Research Group, are available from:

AWSNA Publications
3911 Bannister Road
Fair Oaks, CA 95628
916-961-0927



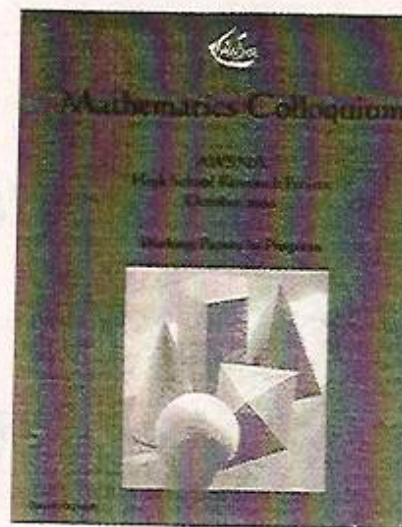
Research Project #1 contains a survey of Waldorf High Schools identifying the best methods to prepare artistic and academic portfolios. There is a list of colleges accepting portfolios, a list of alternatives to college, web sites identifying scholarship possibilities and a section dealing with social/emotional skills—specifically dealing with stress management, sexuality, gender and boundary issues, and conflict resolutions.

— 96 pages



Research Project #2 is the Proceedings of the Waldorf High School Chemistry Colloquium and contains the thoughts of the participants on the chemistry curricula in grades 9-12, the expectations of colleges and universities and several articles on chemistry.

— 186 pages



Research Project #3 contains the thoughts of the participants on the Waldorf mathematics curricula in grades 9-12. Articles include The motives underlying the high school mathematics curriculum; Qualifications for High School Mathematics Teaching; Sculptural, Mathematics and Construction; Chaos Theory for 12th Grade; The Regular Heptagon; Math Difficulties—Diagnosis, Symptoms and Therapy; and others. The colloquia also looked at which capacities were developed through the various math topics, and offers advice to elementary teachers with regard to preparing their students for high school.

— will be available in February 2001



Participants in Mathematics Colloquium:
Front row L to R, Connie Gerwin, Marty Levin, Marisha Plotnik;
Second row L to R, Cynthia Bennett, Bonnie River;
Third row L to R, Ernst Schuberth, David Booth, Harry Kretz;
Top row L to R, Ron Jarman, Erik Marstrander, David Mitchell.

THOUGHTS ON RETURNING TO AN "EDUCATION TOWARDS FREEDOM"

THE VIEW FROM BOTH SIDES OF A WALDORF EDUCATION

by
Mick Follari

It was hot and humid in that classroom, the sky had darkened in the late afternoon with an impending thunderstorm. Outside, the New England trees remained thick, tall and green, shading the grassy courtyard centered among several GeoChem buildings. The classroom/laboratory was old with linoleum tiled floors and ugly fluorescent lighting, with old wooden cabinets around the back and side walls. It smelled of chemistry. I was sitting in on Chem 31 at Brown University during the first week of my freshman year, trying like everyone, to settle on the proper arrangement of courses, which sections I should be attending, etc. As with all my fellow engineering students, the pre-med students, and others, we were observing several different Chemistry courses (and Calculus for that matter, too) in order that we could determine whether we should attend the AP track, the beginner, or some other in-between courses. As the lecture wound down, the professor bid us good day, and as he packed up his audio-visual materials, a line began to form at the front of the room. I could read in his face a stoic resignation as he faced all these young precocious Ivy League students, and it flashed once in a while with impatience. In line, I was mulling over what we had been shown, "photos" of electrons taken by an electron scanning microscope. I was fresh from my Waldorf schooling, and was interested in discussing his presentation since, it seemed to my humble mind, that if you design an apparatus to find little balls, you would certainly find what you were looking for. Through rose-colored glasses, doesn't the world look rosy?

"Yes?" he asked in a tired, preoccupied voice.

"Uh, professor, you know in my courses in school, my teacher said that all this was just a theory..."

"No, we know this now. These pictures are *real*." As if on line at a fast food restaurant having to ask what french fries were, I backed away, slightly embarrassed. I suppose I could have pursued it in his office hours when there would have been time to discuss it further, but I, being stoic, melancholic, half Sicilian, overly philosophical that I am, I let it go.

That moment marked the beginning of a profound shift in my thinking. It was startling at first, but eventually I became quite comfortable in my studies, having set aside the phenomenological beginnings I'd had. In the end, I didn't take his course, I was better placed in the standard beginning Chemistry anyway, and I went on with my engineering studies. The shift came down to this: I decided that in order to survive this program, I would need to 'sell out' to a mechanistic understanding of the world, and put away my still-soft, still-unformed science from Green Meadow. I enjoyed it, at the time; how easy it was to keep abstracting! Like Ukrainian dolls, one opening onto the other, I had mechanistic explanations for everything. I even developed my own mechanistic model of death, after the passing of my father, influenced by my courses. The experiential, participatory nature of science, as practiced with my high school teachers, would need to be stored and held until such a time as I could speak eloquently about it in a context where I understood the mainstream. I would set sail across the sea, through the bleak waters of a soulless study of the mechanics of the world, toward some bonny shore that I

knew lay on the other side, where I could plant the seeds I would guard in my pockets from Green Meadow.

I converted rapidly. Seduction at college takes many forms, and for the Waldorf student, many are classroom-related. Realize that an American Waldorf student feels simultaneously 'special', privy to things not discussed in the public, and somewhat alienated. I think most of us could feel the richness of which we were a part, and yet it was at times uncomfortable. I had many friends outside Green Meadow, and I carried some amount of sheepishness about my school. I had a very private inner life that was nurtured at Green Meadow, but which I would not expose in public. Part of me longed for the normalcy of a big, established, socio-emotionally 'convenient' school, in lieu of the difficult engagement of my own. And I, being of a philosophical, poetic nature, was alternately impressing my new friends or hiding things from them. Thus, at college, convenience of thinking, and the 'freedom to be normal', ironic as that sounds, is a lusty enticement.

Perhaps a little biography...

One of my earliest memories of school, which happens to be a math one, is from the age of 4 at the Montessori school in my northern New Jersey town. I was already an 'upperclassman', at age 4, having started at age 3. I can remember memorizing multiplication tables at the knee-high plastic table they had there—the 7 times table was hard at the time (can you hear Steiner groan!). For about a year, at age 6, each day I did two half days of school, one at Montessori, one at 1st grade in public school; then I continued on in the public school. At about that time, my mother began teaching private viola and violin lessons at the Green Meadow Waldorf School, promptly taking over the music department and creating a much beloved, rigorous and exciting music program there. Sometime in 5th grade, my public school teachers called her in to encourage her to pull me out of the weak public schools in our town. I had visited Green Meadow and loved it, and in my 6th grade year I switched. My class accepted me without hesitation, and I rejoiced in the renaissance atmosphere and continued at Green Meadow through high school, soaking up and breathing in the curriculum of arts, music, sports, science, literature, and math—flexing new capacities every day there. Following high school, I attended Brown University for four years, graduated with my BS, and following a short engineering contract, set out into the world to explore, climb mountains, live, love and experience all that it had to offer. Last year I took up a teaching position at the Shining Mountain Waldorf School, wrestling with a mixture of High School Mathematics and Physics courses.

Math in my high school was fun and interesting, taught by a diverse group of active souls. Quirky Dr. Booth kept us chuckling with stories of the crazy German math professor who drew all his blackboard figures to twisted and incorrect scale so that his students would work abstractly on his problems of rectilinearity and parallel lines. We investigated his newly manufactured Zometool geometrical kits, he described the design and innovative manufacturing process, and I did a little work in Catastrophe Theory as an outside project, unknowing of its relation to the recently unveiled science of Chaos. I can remember the gleam in his eye when he talked of his beloved Smullyen, Bucky Fuller, or Pascal; and we studied logic intensively. I can remember distinctly the long factoring sheet we worked on before Christmas of that 9th grade year. We even survived his famous 10th grade block where the whole Main Lesson Book is written in $67\frac{1}{2}^\circ$ lettering.

Sometimes for a break from Algebra II, Mr. Hoffman would read us stories, hand-written on lined paper, from his forthcoming book about his youth, sort of an upstate New York Huck Finn of the 60's. The stories always involved some kind of small-town, upstate New York trouble-making, and the characters were as familiar to us as those of any movie. He has since published his book printing all the stories

we had beta-tested. We knew of Dr. Edelglass' ongoing work which would become the deceptively slim, dense *Marriage of Science and Sense*, first released in print about two years after we graduated. In 11th grade, two of my compadres and I were 'kicked out' of Algebra II to take up Calculus; and in 12th we worked with Dr. Edelglass in a Physics track class. The teachers joined us for Chorus (perhaps because we had 'gone through' several chorus teachers in those short high school years). In the bass section, I stood shoulder to shoulder with my teachers, or else felt John Wulsin's firm hand on my shoulder from behind me, following some mischief.

I have powerful memories of certain distinct math topics— Logic, Euclidean Geometry, Calculus, Factoring, Imaginary Numbers— and certain blocks, *Parcival*, the Transcendentalists, Ancient Cultures, Comedy and Tragedy, the *Odyssey*, to name some. Besides working, during the summer of 10th grade, the opportunity to go to a national Math and Science camp came up, and I rode the train out to South Hampton, NY to join students from all over the country for three weeks on college campus to study Inductive proofs, competition heuristics and computer programming (in the obscure and useless Prolog language). It was a precursor to the college experience (in and out of the classrooms), and I struggled to understand the abstract math. My fellow students, many from New York City or Los Angeles, were vigorous academics. Many seemed like normal kids, with thriving social lives, and I got along well with a number of them, finding that they were not so different from myself. We were all guarding hidden talents in the light of social demands, and slowly, with time we would let some show. My mother had made me take my cello with me to the dorms, which I was terrified to take out.

11th grade was dominated by *Parcival* and 2nd semester Calculus. During the summer of 11th grade, I drove out to Alfred, NY on my 2 month-old license to spend a week with prospective engineering students doing various mini survey courses in various fields of engineering. The experience was similar to that of the previous year, except my problem-solving capabilities were stronger and I was confident enough to take a leadership role in some of the workshops. In 12th grade we dug deeply into the Transcendentalists— Emerson spoke perhaps most powerfully to me— Russian literature, Physics. When the dust had settled, I had managed to get into Brown University where I studied Civil Engineering with a Focus in Environmental Problems.

That college chemistry class was probably a special case for me, and maybe it touched a raw nerve since the foundations on which it is taught are of an outdated but widely used, very successful atomic model. Though the effect is to leave many college freshmen with a distorted picture of the truth of reality in the form of little orbital balls, by sophomore year in Materials Engineering or in Solid Mechanics, I was comfortable with it. In my other classes, Calculus, Engineering Statics, Dynamics, I had no problems, or I had shifted into a new mindset that would last for four years: "pocket my Waldorf education until I got my degree". I came to really enjoy my math courses, and admittedly, the mechanistic modeling of natural or engineering systems. I was speeding off into the seductive abstraction of a traditional scientific mindset.

These experiences, ranging from being very young in Montessori through public school, Waldorf school, enrichment camps, Brown University, and most importantly, preparing the courses I now teach, represent a dizzying range of mathematical and scientific approaches and circumstances. Though they have all shaped my abilities and experience in math and science, I have returned to a Waldorf classroom, now at the *front* of the room. In my first year, I struggled too much with the newness of being a teacher and with the content of my courses, to really develop them correctly. Perhaps now I am finally in a place where I can really contemplate my courses with some real insight. Sitting with Stephen Edelglass in September (only two months prior to his passing, it would turn out), only a week before my

own 12th Grade Physics block would begin, digging into questions on the nature of light, on seeing, and contemplating how we know what we know – I can see how far he has gone with his work in only ten years since he worked with us.

As we wrestled, as I *wrestled* in conversation with him, chewing on each new piece that he brought– pieces that have not yet appeared in his books– I could feel something deep inside reactivating. Through the sound of his voice and the difficult questions an engagement was reactivating: the heads of the plants grown of those seeds, now ten years sheltered, were poking through the earth, sprouting. And this time I could bring a fuller understanding. I had read a number of books and articles, and I could engage in this discussion in a new and mature way. The concepts of his discussion, combined with the intense problem-solving training and an artistic discernment, would come together to form a confident stance from which to approach my own physics and math blocks. *What a difference!* Like one who, having lived on convenient microwave dinners, returns to the art of cooking real food, preparing it and living into it, I had found again the active, participating, groping force. I can see now the seduction of easy answers; mechanistic thinking has such successes, and the mainstream had bargained me away from that kind of engaged, phenomenologically driven inquiry that I had been part of in school. In the graphic arts and photography, both of which I have pursued since Green Meadow, I had found new eyes with which to see the world, and now, in preparing my math and science courses, I was finding new ones still.

Nothing I have done in all the time since leaving high school has served to bring into focus all the math and science that came before, or cast a light out to the future, as preparing and teaching “Projective Geometry”, and the 12th Grade “Physics of Visual Experience”. Through these, and a deep and active contemplation of the whole program from 9th through 12th, I have seen into my own experience more deeply and suddenly than ever before. In the middle of my busy first year, as the time approached for me to teach Projective Geometry, I worried about it, having had no experience with it. After gathering materials with which to study, I spread my paper on the tables at night at the school, spending hours to complete the drawings.

The experience was at once intense and expansive, concentrated yet relaxing, and also therapeutic. In contrast to the mental tightening in Euclidean geometry, the reductionist thinking of dividing and separating, and of analysis of the parts, this new geometry was stretching and filling. Contrasting and yet complementing the earth element of the former with both an air-like expansion, and water-like mutability, it suddenly offered a kind of epiphany that what I had known before occupied only a small corner of the vast mindspace, was only a rigid special case of the mutable possibilities in a new experience. Add to this experience the contemplative activity, the fire to metamorphose all of it together constantly, and I could feel the reuniting forces at work. The wholes were emerging around me, and the parts, fractured from my reductionist training, were reuniting into the phenomenal entities they had been. Consider carefully the process as I felt it: the drawings originate with some simple instructions, rules that nonetheless allow for a freedom of action in where and how to assemble the figures exactly on the page. Students of mine would respond to the instruction “draw a line”, with “where?”. I have to explain the freedom they have in their actions. Then, slowly, the forms come into focus and develop into beautiful figures. Immediately my reaction was to draw the same one again, exploring the freedom of placement, testing it, and always it would come out correctly, though perhaps twisted, shifted, stretched or the like. In one majestic gesture, I realized that the finished form was a marriage of both the organizing principle and my freedom of action, a lesson I had approached in spiritual studies independent of mathematics. This realization shed new light on all my college studies as well, and I rediscovered a sense of my entry into

engineering from the start. I had wanted to 'know what they know' so that I might help direct the course of evolution by the thoughtful application of freedom in thinking.

And so, as I enter my second year of teaching, still very green and still cautious of the grandeur of my own epiphanies, I can share some of what has become apparent. I was fortunate to inherit the structure of the math program from the venerable Dr. Marty Levin, and though it is quite distinct from my own experiences with Dr. Booth, I have come into real admiration and appreciation of it.

With only one year of teaching behind me, I am far from expert in the curriculum. But I can speak about my impressions relative to my mainstream study since my own Waldorf schooling. As I contemplate each topic, new revelations come to me, and I am able to make conscious the implicit experiences that I had as a student. As a simple example, consider the sine function. With the 11th grade last year, in reviewing plane triangle trigonometry and moving into functions and Analytic Geometry, I thought to present the sine function in the three forms that I regularly used to analyze it, as it were, as the relationship between sides of a plane triangle, as the "shadow" cast on the y-axis of the unit circle, and as a waveform on a cartesian graph, with shifts and stretches. The 10th grader studies Pi and Trigonometry, in the plane, often completing scalene triangle problems to determine the height of a mountain (law of sines). Their understanding of the sine is as the relation between opposite and hypotenuse of a right triangle, a two-dimensional figure. In 11th grade, we take the triangle, fit it into the unit circle, and 'crank' the radius around and around the circle, following the stretching and shrinking of its 'shadow' on the y-axis. This adds a dynamicism, some movement, and a relation between the sines themselves (beyond the fixed sides of one particular triangle). Finally, we imagined that a running paper tape, like a heart monitor, would record the sine value as we rotated the radius, producing the sine waveform, an infinite view of the sines arrayed before us, and from this we get the wave formula complete with stretches and shifts as a collapsed algebraic representation. Here the student experiences something of the gesture of the sine function, existing in several forms, and realizes the connection between the three, without pinning the concept of sine on any one solely. Of course, those who continued would discover the Taylor Series expansion of the trig functions as well. All are ways to *represent* something that is still somehow slightly elusive of any one representation, and yet into which we have gained some insight.

And so, as our own Western scientific history developed dramatically from Plato to Aristotle to Galileo to Descartes to Newton, and on to Einstein and Heisenberg, the students experience a different yet parallel evolution. They progress from the summary in 9th grade, focussing on algebra (lawful balance) and graphing (representation), factoring (decomposition), and probability (capturing the math of games, or chance occurrences) to contrasting and comparing in 10th, so evident in the Mechanics, to analysis and discernment in 11th, to synthesis in 12th, where the Calculus will ask of them to unite their skill forces with a creative, intuitive problem-solving method. How appropriate that even in 9th grade, their experience of what will be called functions (a line, or parabola for example) is to move from the tabular (recording positions of concrete points) to the graphical (plotting), and finally, abstracting those to the algebraic (formula). Therein lies also something of the mainstream scientific method as well. We see how experience can be represented, then abstracted or encapsulated, 'mathematized' and then manipulated for use in solving equations, with the concession that *in the abstraction, we have chosen the outcome*. Pieces of the whole lie in each part, because we have brought the intention of the outcome. This essential *method*, of mathematizing nature for study, is the science begun by Galileo continued by traditional mechanistic physicists, and it is appropriate that this 'earth-element' kind of analysis take place in 9th and 10th grade.

They are also getting a feeling for what will become the cornerstone of Calculus, building up discreteness into (apparent) smoothness. They can feel their growing power to model something directly experienced (tabular) with something abstract (graphical or algebraic). And all the while there lurks the sense that they are connecting discrete parts to form a whole, which then can be expressed abstractly. From here they move on to 10th grade geometry in the plane. Logic, formal proof, Newtonian mechanics, plane trigonometry, the application of math to nature, modeling— all serve to reinforce the comparative, and begin the analytic. At the end of the year, we have a block in growth and exponents, logs, mathematical form and pattern, the relationship of mathematics to nature, Fibonacci sequences and the like. I consider it an exploration into the obscured math in nature, and a chance to develop capacities for recognizing patterns. Here, at the end of 10th grade, in the last math block and in the Mechanics, they get a glimpse of something profound which I hope for them to realize fully by 12th grade: that nature forever remains arrayed around us, a myriad of phenomena, and it is our intentionality, the filters through which we *choose* to see, that expose certain of her special relationships.

In 11th grade, the plane geometry is exploded onto the heavens in spherical geometry, which has still a tether to the earthly, and a nod to the craftsman who beat the academics (John Harrison, the chronometer). They may also wrestle with the difficulty and imperfection of projecting the globe onto a paper plane (a similar experience to their exposure to the archetype in the Pi block). In Projective Geometry they get the humbling yet expansive experience that what they once analyzed so discretely (plane geometry) collapses into a mere slice through the spectrum of 'form'. In this, they experience something like "Goethean mathematics": that the forms with which they thought they were familiar become merely frozen pictures of something larger, more dynamic— a flowing continuum. Their experience of the conic sections is one where in one simultaneously kinesthetic and intellectual moment, the parabola, hyperbola, ellipse, circle, pentagon, and five-pointed star, all unite into one interwoven figure, one experience that they can suddenly stretch, unfold, zoom in on. A moment of epiphany! Here is the start of an incredible insight for the student that the altogether different and distinct pieces they have been scrutinizing, each seemingly separate from one another, shuffling from one class to the next, are actually multiple faces of common things. Forevermore, they will look at say, a hyperbola, as the view of the edges of an ellipse from a seat at infinity, parabolas as stretched ellipses, the ellipse as simultaneously defined by, and defining the five-pointed star and five-sided figure. As the drawings unfold under their hand, they slowly 'get it', and the furrowed brow relaxes, the smile creeps in, and the student can hardly make the last several lines for the rush they are in, the joy of this marriage of hand and head. The governance of form retains something mysterious, and in fact they are feeling that rather than building up a set of conditions with which to fix the figure (plane geometry), they are uncovering windows to a natural mathematics. There is something of water in this process, and yet they also soak up, intuitively, that there was intention behind the conditions for the drawings, and that they could not simply morph into anything from anything else. That although they were free to "draw a line" anywhere, there was an organizing concept behind the drawing to guide it to a predetermined realization of beauty in form.

In Projective Geometry, they work loosely yet rigorously, without mensuration but with intense mental effort, and in this way live into the forms much as we want them to live into phenomena in nature in the sciences. Rather than pinning the forms like dead butterflies for classification (reductionism), they might understand the conditions under which the forms arise, analagous to Dr. Edelglass' goal that they understand the conditions in nature under which phenomena are experienced. So following Projective Geometry, the students are in a powerful seat with which

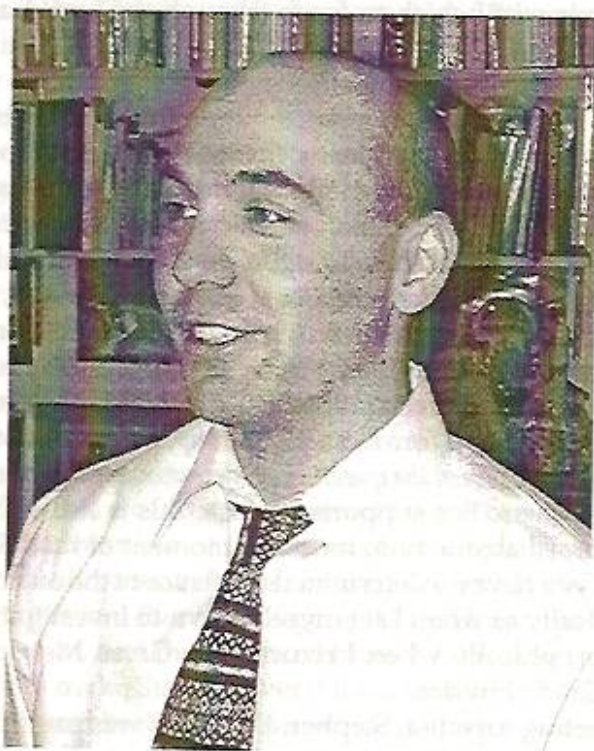
to finally explore “the Physics of Visual Experience” in 12th grade. Here, they will get the complementary of these two uniquely Waldorf courses, where they will bring into further consciousness the contrast between a ‘Goethean’ approach to science with a mechanical one, and perhaps, all at once the artist, scientist, musician, mathematician, poet, and historian will unite in them as they contemplate the phenomena of the world.

But the main goal is not to turn them into strict phenomenologists. They will need to ‘survive’ their University courses probably in much the way that I did—by rigorous ‘earthy’ calculation, proof, study, and some measure of memorization. Excepting those few who dig deeply into modern physics, or those whose College faculties are enlightened Goethean scientists, students will need to stow some of their Waldorf training during the time they gather some conventional credentials. I admit to being guilty of carrying this view of them: trekking through the wilderness of college with some aspects of their Waldorf schooling hidden deep in their backpack, awaiting the space and reactivated interest to take it up. I hope that they will carry with them these seeds of something brighter, that they will carry the view from the infinite, and realize, perhaps, that what they are engaged in will often be a study of the *models*, not the *truths*. If they can realize this, and if we have given them the tools with which to forge through the world as participants, not simply ghosts in the machine, then we will have succeeded in freeing them from one of the prevailing dogmas of our time.

I think of the best that came of my Engineering studies: besides a thorough understanding of the mechanical universe, it is perhaps a truly comprehensive problem-solving spirit and methodology. Though it usually finds its outlet in very mechanical applications, hidden in it is something deeper. I expose for my students very clearly the process that is happening for me when I am solving a problem: to consider carefully what you’re given, and what is being asked, then to assemble a connection of processes between the two using all the tools you have. In this way, before you have put pencil to paper, you have a clear path on which to proceed, with only the ensuing algebraic details to work out. This can be very creative, very deductive—it can have many forms, steps, methods and heuristics. But ultimately, it is clear that *the formulation of the question predetermines the shape of the whole process*, and as much as mathematics supports science, this is itself an important lesson. That from the place of abstraction, from the moment of modeling, at the point of mathematization, you have predetermined the flavor of the outcome. This was never so clear kinesthetically as when I set myself down to investigate the Projective Geometry and philosophically when I returned to Green Meadow, and Stephen to discuss the 12th Grade Physics.

In our last meeting together, Stephen Edelglass was careful to point out to me some of Steiner’s words: that we experience wholeness through the two windows of sense and thinking; and that in thinking is our freedom. This is a feeling that can be carried throughout the mathematics program by realizing that for every problem, the students are exposed to an organizing concept, with varying amounts of actual thinking that progressively increases with their facility in math. Eventually, the student may realize that she is providing the organizing concept and has the freedom of her thinking to interpret the problem and outcome. This will be clear then in the traditional scientific method where sense experience will be analyzed through mathematical abstraction. The successful Waldorf student will be able to understand at what point the sense experience was abstracted, and where the intentionality of a mechanical description was imposed. Here is a hint at the spiritual—that the sense world is permeated by the symbols of spiritual fact. And that we bear the intentionality of future outcomes with us through the processes of living now. There is a metaphor of the spiritual in problem-solving, in which creative, sometimes fluid processing nevertheless unfolds toward an outcome of which there was intentionality at the start.

Several years ago, after I had finished my degree, and had subsequently avoided the field of engineering by traveling widely, I had a profound and beautiful spiritual experience while in Peru. In the following year or two, while investigating all sorts of shamanism, and ancient beliefs, as well as delving into difficult mathematical and scientific studies of Sacred Geometry, modern Physics, and Complexity theory, I found myself trying to develop a spiritual view that would unify all of them; I looked to a hidden order, an unseen organizing principle. Since then, I came to realize the free, powerful organizing power of our own thinking, even our own morality. Careful reflections on the curriculum, even with a limited understanding of Steiner's works, have exposed the themes from my own Waldorf schooling, even intuitively, that are beginning to become conscious. I am careful, however, to hold on to the feelings I have had as a student, and to keep the spirit of the journey my students will make beyond Shining Mountain Waldorf High School in mind as I teach, so that they may carry with them the seeds that they may nurture to growth in their own futures.



Mick Follari is a high school Mathematics and Physics teacher at the Shining Mountain Waldorf School in Boulder, Colorado. He is a graduate of the Green Meadow Waldorf School in Spring Valley, New York, and Brown University in Providence, Rhode Island. Besides his love of teaching he is also an avid musician, writer, and photographer.

Pedagogical Motives for the Third 7 Year Period

by

Erik Marstrander

If one asks a primary Waldorf school teacher what the main purpose or aim of teaching is, many will reply with a theme taken from *The Study of Man*, namely the breathing of the soul. That is, the pendulum motion between the more dissolving and the more crystallizing forces in the soul of the child. The choice of subjects and their content is the tool the teacher has. Of course it is important what children learn, but it is equally important how, and in what order, they learn it. Subjects themselves are tools with which one reaches into the thinking, feeling and willing life of the child.

Should one ask the same question of an upper school teacher, one would notice that the answer becomes less certain, not only because Rudolf Steiner said much less about these grades, but also because our experience (in Norway) is only about 20 years old. It isn't so easy to put one's finger on the pedagogical motivations or aims of this level.

One can quickly come to answers if one considers these grades too narrowly. It is easy to point out that the specific knowledge and skills of the various subjects are the most important element. That might be described as a compromise between a purely anthroposophical system and a public-school system. Discipline of the will could also be described as a central demand teachers place on students. Another point of view might be to see the structure of the single subjects as a synthesis of that subject's limitations and a broader or more general perspective. But these sorts of statements can also be made about the subjects in the primary school. In my opinion, they would only deepen our understanding of content, but not give a *pedagogical key* for our work with young people. That can first be attained when the separate subjects are placed in the perspective of a pedagogical purpose or aim.

The 3-fold principle of form and the first 7 year period

A central concept in the anthroposophical understanding of the human is the concept of 3-foldness. This is a concept found in the physical as well as the soul and spiritual being of man. It is a universal principle of form, which finds its expression in the bodies of animals, the climatic zones of the earth, as well as human soul life. It is actually a

trace of earth's evolution, and every form of life and matter on earth has been touched and formed by this principle. It is ubiquitous. It finds expression in many different ways, and in human existence it may be discovered in many different forms on different levels.

Let us examine the first 7 year period, and see how this principal finds expression there. The human body is the central area of growth in this period. Of course, feeling and thought-life are present, but the strongest expression of the child's existence is in the physical body. In the embryo the central nervous system with the head develops first, and after birth the body grows relatively more than the head. Figuratively speaking the body grows out of the head, and it expands itself into the three-foldness of head, trunk and limbs. It is a physical development in a 3-fold organism. Parents and pre-school pedagogues know intuitively how important it is for the child's further development, that it cultivate and use its body at this age. From the first moments of embryonic development one can follow the physical threefoldness, and it ends with the appearance of the new teeth at 7 years.

The second 7 year period

At this time, the growth forces turn inward, toward the child's inner life, the soul element. Invisible for the physical eye, but very visible for parents and teachers, the child now takes its first steps in the growth and forming of the inner soul forces. Mental picturing, must be exercised and cultivated in a wholly different manner than before. The same holds true for the feeling and willing life. It is very easy to misunderstand the child's progress in these years, and to believe it is important that the child expand and cultivate these forces freely in these years. That however is something which was central to the first period, and now more and more becomes a thing of the past. Pure expansion and cultivation must be led into a motion between the inner and the outer, the upper and the lower. Harmonizing the apparently contrary forces of reflection and willing are now a central pedagogical aim. By nature the child's soul-life is unbalanced, it has a certain nature, also known as a temperament which gives the child both strong and weak sides. This expresses itself in that the child may be over reflective, a dreamer, or more unformed-spontaneous, depending on which of the soul forces is more dominant. Any one-sidedness which isn't harmonized through pedagogical activity at this age will have a lasting influence on the further life of the individual. This harmonizing isn't merely a pedagogical aim, it is in itself a tool to help the

individual develop in a balanced fashion. When the three elements of the soul, thought, feeling and will are harmonized, then one can say that the individual has a 3-fold soul-life. That cannot happen without the assistance of other humans.

Before we examine the third 7 year period, we should look at certain traits of the 12 school years. First one can see that the child goes from a mythical consciousness in the first years to a historical consciousness in the middle years. At puberty the child arrives at a modern consciousness. This is visible in that when the child is sexually mature, the subject of history is the modern world. Then one can rightly ask: Why 3 more years of school? To which sort of consciousness should we address ourselves, when the child has become a modern or present-day human?

Another question which may be raised concerns the picture or representation of consciousness of the child. The reflective nature of this consciousness bears with it a memory of collective human development. Greek culture isn't simply a theme for the sixth grade, but is also a picture or reflection of what the child's condition is, spiritually spoken. At that age children are *Greeks* in their soul life. The next year new cultures are presented, and that way the child passes through mankind's cultural development in the course of primary school. A teacher can exercise this epicure consciousness by presenting and working through human cultural development in a certain order. However, this guidance ends when the child is led to our time, at the end of primary school. Which aim or purpose should teaching have in the upper school when the reflections and the repetitions of the past are completed?

The third 7 year period

Many things indicate that the high school, seen as a whole, is an exercise for the future. The soul-life's time as the central aspect of human development is over, and the harmonizing of thought, feeling and will should be well under way. Along with the specific aspects of subjects mentioned above, everything points towards the necessity of finding a new point of departure and purpose in teaching after puberty. The threefold principle has moved from the physical development since birth, through the harmonizing of the soul, and now approaches human spiritual life. This life also expresses itself in 3 different ways. We may call them imagination, inspiration and intuition. These we must now examine more closely, to determine their nature.

First, what does it mean to imagine? Inner pictures after puberty are totally different than the

spontaneous inner pictures the child has. The post puberty pictures must be exercised consciously, which means, the will must enter the thought realm, to keep the thoughts from simply flitting about. Without this effort thoughts and pictures will be dominated by their outer forms, and not be created by an inner content. An exercise many teachers recommend, and which comes from geometry in the 9-10th grade is the following: The teacher describes a body which the pupils can't see, but which they try to picture. For example: From one angle this body looks like a square, from another like a hexagon, from a third like rectangle. (Fig 1) Looking at these figures one quickly sees the body is a cube. Without them, however, it is much more difficult. Each of the pictures is a correct, but incomplete picture of the cube. The imagination of a cube first becomes clear when these three, and many other pictures are put together. The imagination is gradually built up of single pictures. They are then put together to a whole which tells us more than each single pictures.

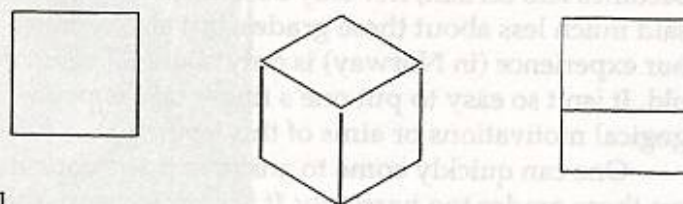


Fig 1

The will is used to produce as many images as one can of the body and to hold them together. Figuratively speaking one sees a thought from many different perspectives. The truth isn't in each of them, and not in the imagination either. But imaginations give humans a certain freedom. A single picture can be compared to approaching a problem from one point of view, while a fully developed imagination gives the freedom to look at and judge the problem from many different points of view. Even though this example is taken from geometry, one can call upon the same capacity in social sciences, drawing, mathematics and natural sciences. Imaginations develop when one consciously uses ones will to approach a problem from different perspectives. That, I believe, can be done in all subjects.

The next word, inspiration, points to something in the feeling life. A good example might be in theater or stage work or eurythmy. One must practice certain activities or sequences again and again. Often this will seem like a long walk through the dessert. The will drives one along, but is one capable of filling the exercise with feeling? The teacher may make suggestions, then the pupil has to try again, to

mobilize the inner forces necessary for the exercise. This may go on again and again until one suddenly hears: "That was it! The way you did it now was convincing!" At such a moment one can experience an inspiration. The first time it might be difficult to catch sight of it, but once one's attention is drawn to it, it becomes unmistakable. Repetition has allowed a feeling to mature, and then lifted the whole thing up. Through effort and practice the feeling life is made a vessel for a super-ego. One has consciously worked to create a specific feeling. An interesting point many notice is that success wasn't possible when one strained and worked hardest, it often came at more relaxed, but very conscious moments. This example is taken from theater or eurythmy, but one can recognize the element of practice whether the subject is theoretical, artistic or practical. Practice creates a surplus or overflow in the inner life, and if one works consciously with one's feelings, the work itself may be carried in an inspired manner.

Now, intuition. This word is actually used in two ways. One way is that one has an intuition. There one generally means a thought, a sudden understanding, or an idea. It is often used when one has worked a long time with something without succeeding at it, and then one suddenly gets an idea. On the other hand we often speak of acting intuitively. With that one means the right action at the right moment. I use the word in this sense. An intuitive action is exactly fitted with regard to time, place and persons involved, and is often critical to saving or helping a situation. Often such an action brings more than just itself. Acting intuitively is an art-form, but it is also the result or gift of earlier work. Such things don't come by themselves. To do the right action demands long preparation, for instance that one has thought intensely about a person. When one then meets that person, one is prepared. Then something can happen which hadn't happened before. Intuitive acts also occur in teaching situations. In drawing class a student may for example draw a line which is correct, beautiful and relaxed. Without trembling and straining he can guide the pencil on the paper, and allow the line to acquire an artistic quality. This is of course only possible after long and intense practice. This sort of intuitive act is not reserved for artistic subjects, it may just as easily appear in foreign language class. Only after a long and continuous effort in a language will the intuitive sense of a language appear. The same may be said for any other subject. Only after long practice, for most of us many years, will the will-forces be able to concentrate themselves for longer periods on a piece of work.

The first step is to repeat and practice as for an inspiration. However, to create an intuition one must have worked on and lived with something for a good while. Interruptions force one to begin to practice and repeat again. When a young person has achieved this, in whatever subject, he will begin to carry himself, to be spiritually independent. The pupil knows what is necessary, enters into assignments easily, and creates his own motivation. At this level the young person is working with his intuitive powers.

I have tried to present each of these three capacities as concretely as possible. Seen collectively what they have in common is that the forces of the will are applied consciously in order to develop a faculty. When humans direct the forces of their will into the different realms of the soul-life, spiritual faculties appear. Presented systematically it might look like this:

Exercise		Aim	
Imagination	will forces in thought	picturing	varied pictorial faculty
Inspiration	will forces in feeling	empathy	expansion of the feeling life
Intuition	will forces in the will-life	presence	continuous consciousness/awareness

The fourth 7 year period

These faculties: imagination, inspiration and intuition can first really develop freely after the birth of the ego, around the age of 21. Then the individual has the possibility of developing a meditative life or consciousness. I now want to describe certain meditations, and try to show that the same qualities are demanded in them, as were developed in the pedagogical situations I described.

One of the first steps in developing a meditative life is to develop a many-sided pictorial consciousness, one which is flexible and is filled with warm sympathy for one's mental pictures. To practice this one can take any mental picture and build it up towards an imagination. If one constantly attaches new, related pictures to the original one, one will build up a mobile, many-sided series of pictures. What ties them together is one's own will power. This exercise has many of the same qualities of that which we do with pupils, but there is one important difference: the adult does this voluntarily. One creates one's own motivations and aims, how long and to what degree one practices is entirely up to oneself. The work is totally one's free choice.

Further meditative work then naturally leads one to working directly with one's feeling life, but of

course the ability to feel only what one wants to feel is more easily said than done. In everyday life our feelings come and go, in meditation they must be brought forth intentionally. One can work with one's most affectionate feelings if one pictures for example a growing plant, trying over and over to lead one's warm feelings towards the picture. One comes close to the plant, lets it grow and bloom, go to seed and all the while accompanying the pictures with feelings. The picture must be more than a dry reproduction, but even so the repetitions may be a true test of patience. One must constantly work to overcome the usual lack of interest we have in our mental pictures. Finally one may experience that the picture shines of itself.

The third step is to overcome the uncertainty which the experience of an imagination may awaken. Fear closes one to new experiences. If on the other hand one loses interest, one immediately falls back into an everyday consciousness which is incompatible with experiencing an imagination. If one loses patience and tries to pressure the imagination into revealing more of itself, then the necessary concentration will also disappear. Both too much and too little will power are obstructions to progress, because the last step in this exercise is to strive for a continuous presence in one's thoughts, feelings and will.

This description of three steps in meditation is short and somewhat simplified, but it shows the similarities between what we try to accomplish in our pedagogical work in the upper school, and meditation. What adults practice in a free and independent manner, may be present as a seed in young people. This does not mean that one should introduce meditation in school, that would be a fatal misunderstanding. On the other hand the young people are on a path searching for independence, something which will almost inevitably lead them to working with the aforementioned faculties. This work cannot happen in a vacuum, it is necessary to have material, in the form of subjects in order to show results. That is why capacities and knowledge are necessary at this age. Knowledge, understanding and subjects are however not the aim, they are tools we employ to work pedagogically. We will succeed in helping young people develop an independent inner life to the degree that we induce them to carefully and consciously apply their faculties to their schoolwork. Imagination, inspiration and intuition will become visible for the student and the teacher. That is what we might call a pedagogical purpose or aim in the upper school.

Does this mean we must radically alter all teaching in the upper school? I don't think so. On the contrary, many teachers will recognize these thoughts as quite close to present teaching practice in the upper school. At the most it might demand a change of emphasis. If one accepts working towards a future consciousness as a pedagogical aim for the upper school, then specific questions of subjects must be subordinated to that. It is important that the young people be allowed to have their own experiences in these areas. This might be difficult if all students must take all subjects, as our system is today. We should give more room for individual needs and interests. A more varied schedule, with more opportunities for individual study need not then stand in opposition to our pedagogical aims for the upper school, but rather encourage the students to experience the possibilities for development in an active life. The three levels are equally present in crafts as in chemistry. Some students glide easily into this work. They quickly acquire an experience of their independence, and so take the responsibility of guiding themselves. The teacher may inspire, but they work independently. Others strive hard to grasp their own will, and still others never achieve it in the course of their school years. These faculties do not have the same inner necessity that repetition of earlier cultural epochs have, but are faculties of the future, equally demanding for young and old.

The metamorphosis of the will

The will has here been presented as if it were something obvious or self evident. In the middle of a description of a difficult process, the will and its expressions throughout life have been presented as something stable, something unchanging. Obviously human life is not such. Everything connected with human existence undergoes change in the course of a life.

The process I have tried to describe is how after puberty the possibility for an independent individuality opens up. This may be called the motive or incentive of the will. But the changes in the life of the will not only demand a new schedule, but also that all activity in the school and the way of considering school, indeed that the school is placed in a social context, something which is hardly necessary for primary school. Therefore it is necessary to look at how the changing impulses influence the life of the will.

The impulses of the will work outwardly in a powerful, yet unconscious way in the first 7 year period. These impulses can be so strong that many

adults find it impossible or very difficult to resist children. A child can often dominate a whole family if there is no resistance. Self certainty gets blown up like a balloon, self-assertion grows enormously as empathy shrinks correspondingly. The direction of the will impulses also seem random for the child. Actions are momentary whims, and like a blustery shifting wind, not capable of effecting long term results, everything must happen immediately. The child's will-life seems chaotic. If one however looks to the development of the physical body the very opposite is the case, Here a very powerful and directed impulse of the will is visible. The body and the limbs, with the center in the central nervous system grow and grow, and every little detail shows great wisdom. The soul-life of the child on the other hand seems to be controlled by the physical development, that is in constant growth. This condition of the will's development may be illustrated by a geometrical picture. (Fig 2)

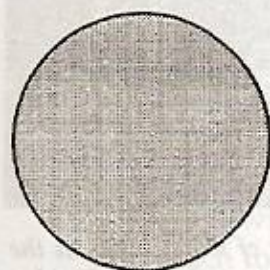


Fig 2

The surface is completely filled up with points. The growth of the circle is caused by more points on the inside, their number causes the circle to expand. The origin of growth is in the center of the circle, and it can grow to become infinitely large.

Most people will sooner or later feel that the random impulses of the child's will have a negative effect, both on the indi-

vidual and the community at large. The young person's will must be dealt with differently at school age than at pre-school age. The work of harmonizing forces must begin. This must again be modified after puberty. At this level individuals begin to unite in freely chosen idea-communities, where each individual's contribution is placed into a larger unity. Generally speaking some sort of system or systematic coherence must exist in order that the individual will or contribution can unite with others. The well known youthful wish to do something is an expression of this. This is a healthy trait. The will forces of the adult also seek a larger system or framework, even the most greedy capitalist who considers himself alone will serve both himself and the capitalistic system. Rudolf Steiner's suggestion, that workshops and craft/trade exercises in the upper school be *outside* the school can be understood in this framework. In young people the

individual will forces look for each other in order to create something. This may also be illustrated by a picture from geometry. (Fig 3) All of the straight lines create a circle by being tangents. This way of making a circle can give an idea of how the will forces work at this age. In Fig 2 the forces stream from the inside out, in Fig 3 they stream from the outside in. An open space is created, and held by the lines streaming from the outside. Teenagers, young people after puberty are striving to use their will in this way. It isn't enough to stand alone and will something. This will wants to flow into coherent surroundings.

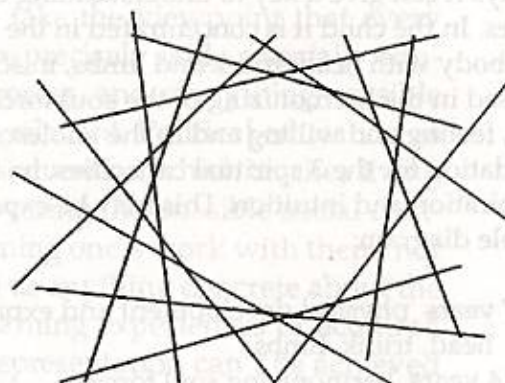


Fig 3

Adult's will generally moves between both of these figures in situations demanding that they work together. These 2 different ways the will forces have of expressing themselves become a social reality. Some people will strongly stand forth as individuals. They incline towards self-assertion, and crowd out others. Others will want to be helpers, to create a place where others may participate etc., they incline towards disappearing. Most of us will move between these two poles, often several times a day. An individual may be very dominating in some situations, and very restrained in others. These expressions of the will actually work together and we need both kinds, both in respect to ourselves and to others. The real question teachers, as well as parents, persons working in the health sector or priests meet, is to what degree are you able to create a space for another person. Children, patients, all humans need others to create a space in which they may grow. The truly non-egoistic use of will-forces is to set them into a coherent unity or whole.

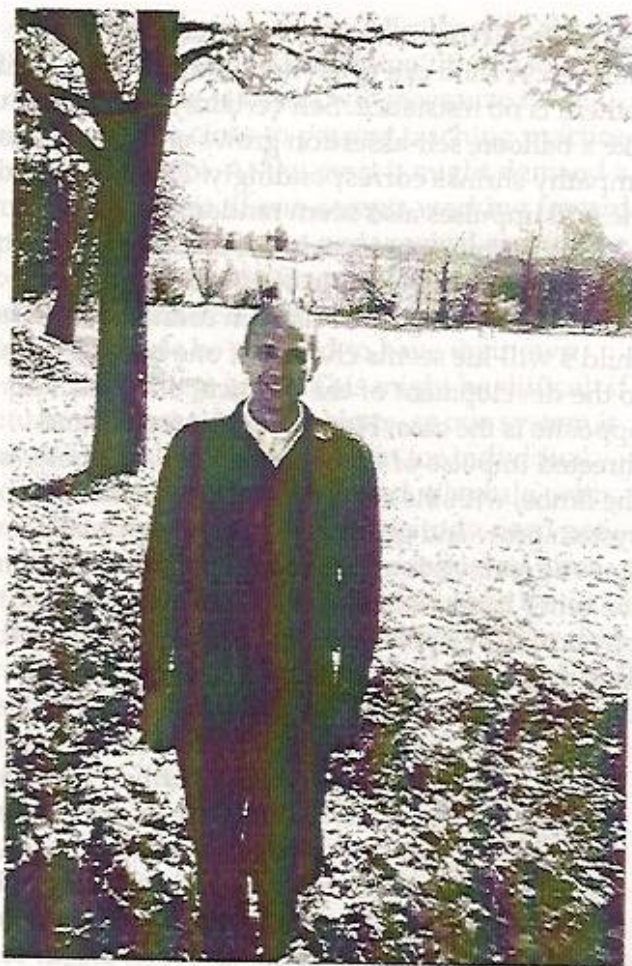
Summary

Many things indicate that the upper school is actually a practice field for the forces of the future. Specifically that means to exercise ones will. Will forces are the unshaped future, as opposed to the more reflective qualities of thinking. Our modern world has been seduced by the magic world of thought, and it is easy to believe that exercising thought is the purpose of an upper school. I hope I have been able to lay some weight on the other side of the scale.

The development of the 3-fold human goes through many phases in life and is a huge field. In many ways it can give a key to understanding different phases. In the child it is concentrated in the physical body with head, trunk and limbs, in school it is focused in the harmonizing of the soul forces of thinking, feeling and willing and in the adolescent the foundation for the 3 spiritual capacities: imagination, inspiration and intuition. This may be expressed in a simple diagram:

- 0 – 7 years physical development and expansion
head, trunk, limbs
- 7 – 14 years harmonizing soul forces
thinking, feeling, willing
- 14 – 21 years grounding of spiritual capacities
imagination, inspiration, intuition

Each phase has a pedagogical aim or motive. To train or create the future through the will, however contradictory that may sound, is a central aim for the time after puberty. Young people are however always seeking a coherent whole, their efforts shouldn't just serve the isolated individual.



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The editors would like to dedicate this edition of the Waldorf Science Newsletter to the memory of our friend and colleague Dr. Stephen Edelglass, from the Green Meadow Waldorf School in Spring Valley, New York. Stephen's love of phenomenology and devotion to truth is an inspiration to us all.

Social Education through Mathematics Lessons

by

Ernst Schubert

Introduction

The connection between mathematics and social education can be examined from various aspects and on different levels. With H. Jungwirth (1986) as well as P. Heintel (1977) I would like to take the viewpoint that every learning activity includes a political - or, more precisely said - a socially educative dimension. All learning is a social process, and everything sensible which has been learned can become socially relevant. Political education as a whole appears as one part of societal-social education which teaches living and working together. It is a matter of discovering the possible social components of education and of consciously forming one's work with them, not of inventing them. Of course this doesn't tell us anything concrete about the significance of specific learning contents, learning experiences or acquired abilities for society. A complete systematic representation can't be achieved here, nor will it be attempted. A number of concrete teaching examples will be sketched from the point of view of social education, but the discussion of each will be incomplete, so that other points of view will remain possible for each topic; other subjects besides math would be able to show, using their means, similar socializing effects.

To clarify where the point of view expressed here has its emphasis, let me briefly note some other possible aspects of our subject. Social ability must ultimately manifest itself in action. In relation to the soul's abilities - to understanding, to emotional relationships and to willingness and readiness to act - many paths can be shown which lead to this goal in normal class teaching. Each of these paths has its own value and each needs the others to complement it. Actions which show concern for others can be practiced directly; in relations with classmates, in sensible social projects, in the role model of the teacher. Mathematics classes, for example, can cultivate a climate of consideration for differing levels of ability in which there is mutual help rather than competition. The emotional relationships of the students among themselves and toward the rest of their social and natural environment, depend to a great extent on how they experience their environment. The concept of Man and of the World which is nurtured in Waldorf Schools tries to show children how the kingdoms of nature are connected to the human being in a way that will influence their feelings and emotions positively. Beginning with legends about plants and animals, continuing in farming and on to the natural sciences, as described by Buck and von Mackensen, these goals are like a thread winding its way through Waldorf schooling.

Connected with these paths toward social education there is the growth of the ability to acquire knowledge and understanding. These reflect back on one's emotional relationships to his or her environment and bring forth a

readiness for social actions. Math classes offer a wide spectrum of methods in this context; from the function of mathematical methods in a modern society, for example in statistics, to the effects that computation of interest rates have on the living conditions in the Third World, and on the cultivation of certain ways of thinking through mathematics.

The main emphasis here will be: for which socially relevant ways of thinking can math create a disposition? And how can math, at least partly, teach for life? None of this intends to underestimate the significance of concrete socially or politically relevant subjects or of the importance of teaching methods and interaction among students, teachers and students, and teachers among themselves. The path being followed here must leave many questions open. But the following may direct our attention to an important field of study. So let us begin.

What is Twelve?

As we see it, the possibility of continuing a natural sequence of numbers endlessly can not always be taken for granted in so-called primitive cultures. After a certain number, "many" begins. For example, whoever has no concept of 5 can only think of $4 + 1$ as "many." The recursive definition of each number as the successor of the preceding number, the axiomatic description which Peano gave the natural numbers, doesn't lead to the formation of a concept. The addition of the first numbers is impossible and leads only to the "many." But in reverse, whoever has a concept of 5 can divide this into $4 + 1$.

The Swiss mathematician and logician Paul Finsler (1894 - 1970) emphasized this standpoint in the set theory: if a set is known, the subsets can be identified. The other way around, there is no guarantee that given sets can be unified to make one set. (Finsler 1975).

In his book *A Theory of Knowledge Based on Goethe's World Conception* (Steiner 1980, page 75) Rudolf Steiner (1861 - 1925) gave a philosophical presentation of the relationship of the whole, as represented in a number, to its parts or elements in the context of his discussion of Kant: "The mathematical unit which lies at the basis of number is not primary. The primary thing is the magnitude, which is a certain number of repetitions of the unit. I must assume a magnitude when I speak of a unit. The unit is an image created by our intellect which separates it from a totality just as it separates effect from cause, substances from their attributes. When I think $7 + 5$, I really hold 12 mathematical units in mind, only not all at once but separated into two parts. If I think the whole quantity of mathematical elements at once, it is the same thing. This identity I express in the proposition $7 + 5 = 12$. All judgments, in so far as the members which enter into the judgment are concepts, are nothing more than the reunifying of that which the intellect has divided. The interconnection comes to light as soon as one enters into the content of the intellectual concepts."

This theoretical remark is out of its context here, and of course it can only be fully interpreted within its context. As Steiner himself said later on, a way of thinking is characterized here in which, in view of a complex of individual parts or processes working together, the function of the individual parts are defined by the purpose or function of the whole and their place within it. He calls this way of thinking "analytical," because the train of

thought goes from the whole to the parts. Just as the parts of a car are only understandable through the idea of the car, analytical thinking emphasizes the primacy of the whole, and explains the parts according to their function in reference to this. Steiner calls the reverse thinking process "synthetic," which tries to define the whole through the function of the parts.

In science as in daily life, both directions of thought are certainly possible and sensible. If, for example, we look at the lottery betting tickets with the choice of numbers on them, the result is a total number of choices which can be statistically described, but the single bet can't possibly be defined out of the total number of bets. On the other hand, even the simplest case of reading a number with several digits presumes a definition based on the whole: only in reference to the whole can each part of an organism make sense and be understood.

This difference between putting questions in either a synthesizing or analyzing way can be practiced in beginning arithmetic classes in the simplest but most basic way.

When children learn arithmetic in a Waldorf school, they start out with the concept of a unity in the sense of a set – not in the sense of individual elements as understood by Rudolf Steiner in the text just quoted. The teacher shows, for example, how the hand has five fingers, how the body as seen from the outside has two sides working together – hands, feet, ears, etc. The mathematical operations arise as divisions or separations within the given number. From the beginning the children are shown how 12 can be divided into $7 + 5$ or $3 + 8 + 1$ etc. A very imprecise but nice question to ask the children is: What is the most beautiful 12? Perhaps one child says $12 = 6 + 6$. Some children find that lacking in variety and prefer $12 = 4 + 2 + 6$. An especially clever child may say $12 = 12 + 0$. Or, some like $12 = 3 + 4 + 5$ or $12 = 1 + 2 + 3 + 3 + 2 + 1$.

The twelve remains unchanged in all these answers. It is the whole, that which stands above and holds all the answers together. But how many answers are there to the question of twelve? Not all are correct. Perhaps a child says $12 = 7 + 6$, which is wrong. One cannot answer just anything, and yet, there is a difference to the usual question of: How much is $7 + 5$? The teacher knows the answer of course, and he is just the referee who says "right" or "wrong". For him it is actually a rhetorical question. But when the teacher asks, "What is twelve?" or "What is the most beautiful twelve?" each child who answers gives him a problem to solve. The child learns from a formal example, half unconsciously, that a question can have many correct answers but that not just anything is correct.

In an interesting way analyzing and synthesizing thinking processes correspond respectively to the development of organisms or technical products. Every organism is, in each stage of its development, a whole, adapted to the living conditions peculiar to each phase of growth, as Maturana and Varela say an "Autopoietical Shape" (Maturana/Varela 1987). In this way, a chestnut is a whole plant in a very special condition. The development of organisms is the result of inner differentiation or functional specialization of its parts. On the other hand, every technical product must be synthesized from its parts. The production process is largely a putting together of separately produced parts. Probably a technical, engineering oriented construction of a human being would start with a static frame, the skeleton, and add

the other parts one by one. But with organisms it is hardly ever a matter of simply adding substances or parts, rather it is assimilating them into their life processes.

Many trains of thought could be developed out of this. One of them is the question posed by R. Fischer (1984) in connection with the computer when he asks: "The question is, whether a different, more flexible mathematics is possibly one which is more appropriate for work with living organisms."

Remarks

The modes of thinking referred to here could lead to misunderstandings. Could it be that these possibilities of our thinking processes are a metaphorical use - or misuse - of mathematics? Actually, one can differentiate between different styles of thinking completely independent of any special contents. Yet our manner of thinking determines our view of things and processes. Whether such thought processes can be practiced in math, and if they can, whether they can be transferred out of math classes, could be studied empirically. But first this must be discussed at the conceptual level to find the correct questions and approaches. All empirical research needs guidelines, and without them questions can't be correctly formulated and results can't be interpreted. This problem should be mentioned but we won't go into it here.

But that forms of thinking, as expressed here, are socially highly effective, can be easily understood: Doesn't the tendency of our political bodies to give only one answer to every question — which is of course in keeping with their political stance — reflect the way one has learned to solve math problems? We are aware that a problem such as unemployment requires the inclusion of a number of aspects if it is to be completely analyzed. But the discussions in political meetings seldom reflect this. Above all the tendency to see every view of the problem which differs from one's own as competitive rather than as complementary, seems to prevail. If the child taking its first systematic steps in acquiring knowledge becomes accustomed to finding many possible answers to a question, answers which recognize individual characteristics and viewpoints, this would certainly be helpful for the above mentioned situations. The seemingly trite question: What is twelve? can only be answered in depth through a number of different answers. We know the twelve well if we have experienced many answers. Is it any different with our social and political questions? Without overly straining this analogy, in thinking through this simple arithmetic example exactly, we can find the synthesis of the apparent dichotomy between what is valid in general and what is an individual point of view. I cannot know beforehand, which answer a child will give. Surprises and new aspects continue to come up. Once the child has given its answer, the general judgment of it is asked for: Not everything which is said is correct. What is wrong is left by the wayside and what is right is placed, like a mosaic stone, in the overall view of the problem.

But in reverse, the very set, rigid answer which the question: How much is $5 + 7$? requires, is often experienced like a straight-jacket. And how often is it said, in discussions about changing the present conditions, that something just is that way, nothing can be done about it, it's like $1 + 1 = 2$! What a

terrible subject must math be, that the general populace uses it as a paradigmatic metaphor for compulsions! Have our methods of teaching beginning math possibly had wider and deeper effects than we were aware of up till now? In any case, children who have become acquainted with arithmetic in an analytical way will hardly use it as a metaphor for compulsive, straight-jacket thinking.

It should be added, that starting from the unit as the whole, as Waldorf schools do, has a long cultural and philosophical tradition. Many references to this can be found in von Franz (1970).

Triangles and Configurations

Triangles are usually represented as in figure 1. Through projective transformations they can be changed, on the real or projective plane, to other forms such as in figure 2. If we realize this and draw the triangle as a configuration of full straight lines (figure 3), we can see a total of four triangles in which the two parts of 1, 2, and 3 are connected by going through infinity. The original, isolated figure in a plane (figure 1) becomes a configuration of that plane. The isolated triangle appears as part of a totality.

Does this way of teaching geometry, working on these differences, mean anything in relation to the modes of thinking that a child develops? I have often asked students this question when they were dealing with the various forms of triangles at the beginning of projective geometry. Approximately the following was presented: The triangle in figure 1 is a figure in the plane. One could cut it out. The triangles in figure 3 give a pattern of the entire plane. Therefore we can call the pattern a configuration – even if this expression is sometimes used in a different sense in mathematics.

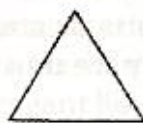


Figure 1

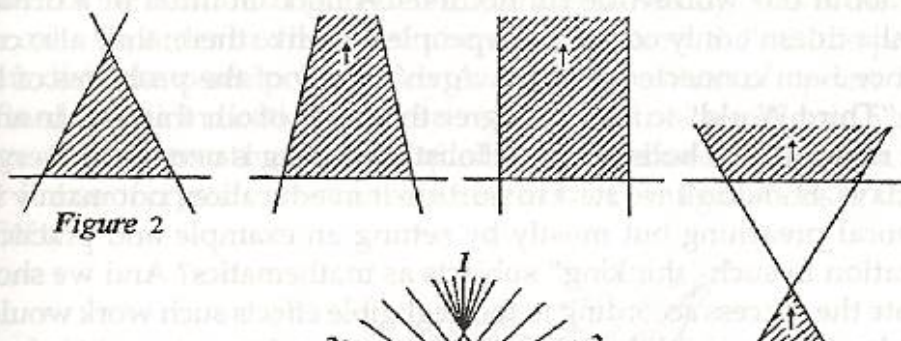


Figure 2

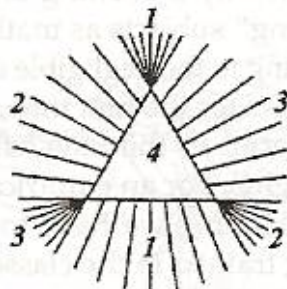


Figure 3

Then the students added: A child who will become a biologist some day could be stimulated to a variety of ways of looking at things. The closed triangle doesn't point beyond itself. That could bring about an inclination to stop at what is given, even with living organisms, for example plants, and to look for the conditions for their shape only in their genes, etc. The configuration, which has to include the huge, infinite whole, could awaken an inclination to connect the plant to a whole. The specific form of each plant reflects not only its genus but also where it is growing, the plants growing around it, etc. A plant can only really grow properly within this entire context. How differently does a dandelion grow, depending on the sun, rain, dampness, dryness, sand, loam, clay or whatever else is in its environment! He who has learned to think in configurations will say: I only understand the single object when I can put it in its place in the whole. Without the germ with its cell structure the plant won't grow, but without the whole environment around it, it won't grow either. Thinking becomes non-real, isolated, when it is cut off from this environment.

Let us try to apply this way of thinking to society. The closed triangle could awaken a tendency to base life's possibilities on one's own accomplishments, income etc. The answer to the question: What do you live from? could be: from my income. The configuration might awaken a tendency to a different view. As a consumer in the modern global economy we are connected to a great number of people whose products and work we use. Even our clothes have often been put together on several continents. Every consumer, child or adult, sick or healthy person, employed or unemployed, is a place where many threads of human endeavor come together, a "hollow" where the work of many people is used up but where every one of us may also contribute something to the whole. Socially, we really can't define ourselves as Mr. or Mrs. X with this or that income from which we derive certain claims or demands. We are always, as long as we consume, woven into the whole which carries and supports us and takes up our work. To correctly define our place in world society we must keep in mind that we are woven into the whole of humanity.

One of the advantages of modern communication is that we can actually know about our worldwide connections. A flood in India or a drought in Australia doesn't only concern the people who live there; they also concern me, since I am connected to them. Aren't many of the problems of the so-called "Third World" to a large degree the result of our thinking in an isolationist rather than a holistic way? Holistic thinking is necessary everywhere these days. Shouldn't we start to nurture it in education, not mainly in useless moral preaching but mostly by setting an example and practicing its application in such "thinking" subjects as mathematics? And we shouldn't evaluate the success according to the negligible effects such work would have on adults. What a child thinks for the first time, when a thought is fresh, has a strong impact on how it forms its thinking habits, though adults may only see an aperçu in such thoughts. For an empirical research into the results of practicing such forms of thinking we have to be sure that these forms of thinking are actually being trained in the classes where we do the research. Such research could be of great value.

Polarities

Our predominantly isolationist thinking, which tries to explain the whole as the working together of the smallest parts – without being able to do this with living organisms – has its most clear expression in atomic theory. Thinking in terms of atoms isn't trained mainly by physics or chemistry teaching, but by mathematics. In geometry this kind of thinking can be seen in a tendency to consider the points as geometric "atoms". Logically over the past 100 years, geometry has gotten used to using other starting points – circles in circle geometry, lines in line geometry, etc. The various systems of axioms are the formal expression of this.

But more important than the logical possibilities given by the various systems of axioms is the training of the ability to think and imagine geometrically. This leads to various forms of thinking. In connection with projective geometry, the discussion of polarities or dual formations has proved to be very helpful. Here, identical structures describe very different contents, which, when they are applied to reality, can lead to completely new viewpoints. (Adams 1965, Adams/Whicher 1979, Locher-Ernst 1970 among others.)

I hope you will accept my using a very abbreviated representation of an example that the non-mathematician may skip over. If you are explaining the pole-polar-relationships of a circle or a sphere (in which a point is related to its polar plane and vice versa) in the projective geometry of let's say, class 11, you can explain that besides logically looking for the causes in a point, you can also look for them in the periphery. What I mean is, usually atoms are considered to be the origin of central forces from which macroscopic formations are derived. To think in a polar sense would mean to ask about the peripheral forces. Crystallography, deriving the form of a crystal out of "atoms," elementary bodies, was begun not because experimental research results were on hand, but as a result of accepted philosophical paradigms. If we apply the polarity laws consistently, the form of a cube, with its regular construction, can be derived from the harmonic figure in the plane at infinity. Figure 4 shows the relations meant here. M- is the picture of the plane at infinity, with the harmonic basic figure from which the cube is derived. The 13 points of the harmonic figure determine the connecting lines of the 8 points of the cube. In order to draw the cube we must know the middle point M+ and the distance to one point which gives us the position of a point. The rest is defined by the indications we can read in the drawing. Figure 5 gives us George Adams' construction of a quadrilateral-hexaedric type of a crystal. Other forms of crystals are connected with other configurations in the plane at infinity.

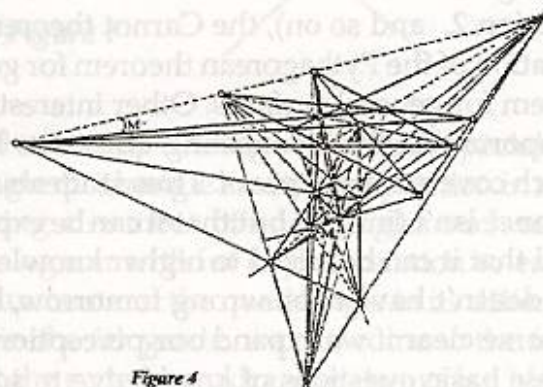
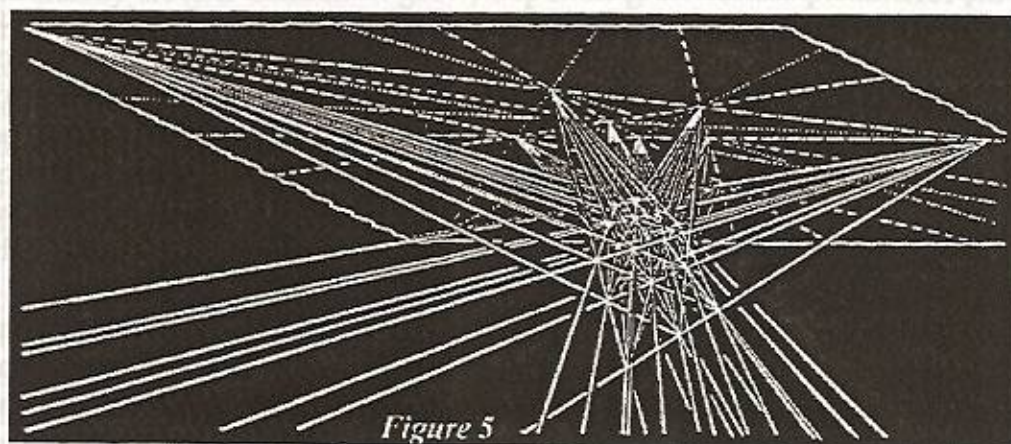


Figure 4

The point here is not to describe physical relationships but rather to point out logical possibilities in which the structure in the periphery determines a sub-structure. For the example I chose, the polar aspect may seem unimportant from the point of view of today's thinking. But in modern physics there are enough phenomena for whose description the use of this way of looking at things could be helpful. In addition, let us remember that our geometric-spatial representation of atomic physics can be replaced by others with the same mathematical structure (Gschwind 1977, 1979, 1986).



The social relevance of a way of thinking which can change atomism to its opposite has already been touched on: Don't social understanding and ecological thinking desperately need paradigms which, instead of constructing complexes out of social or biological atoms, teach thinking of and defining the single parts in reference to their surroundings?

Learning in context

The attempt to structure a number of topics under a common heading isn't specific only to Waldorf education, but the following examples should remind us of this attempt.

Example 1

The Pythagorean theorem is taken up from the 5th or 6th grade on in Waldorf schools. Every year new points of view are added. The main points of focus are: The Pythagorean theorem for isosceles right angled triangles, the general Pythagorean theorem, the Pythagorean theorem for similar figures above the triangle's sides, use of the theorem for calculating lengths (EINBETTEN Equation 2, and so on), the Carnot theorem (= Cosines theorem) as a generalization of the Pythagorean theorem for general triangles, the Pythagorean theorem for general surfaces. Other interesting points could be added. What is important is that connecting questions lead through a curriculum spiral which covers several years. Thus students can learn that what they have learned once isn't finished but that it can be expanded, that knowledge is relative and that it can be raised to higher knowledge. What we have understood today doesn't have to be wrong tomorrow, but the relativity of its validity can become clear if we expand our perception tomorrow. Is there a transfer from these basic questions of knowledge to social understanding

and judgment, or are these fields too far apart? Could subjects like English or social studies take up these modes of thinking, after they have been handled in math, and help them be transferred? And vice versa, could they show how political and philosophical dogmatism have affected mankind? A multitude of questions and research possibilities could arise here.

Example 2

If we put the most important quadrangles into the (already known) "House of Quadrangles" in the order of their symmetry, the most symmetric form, the square, will be at the top. Through a loss of symmetry, the other quadrangles can be described and put in order. The general quadrangle has lost all its symmetry, as in figure 6.

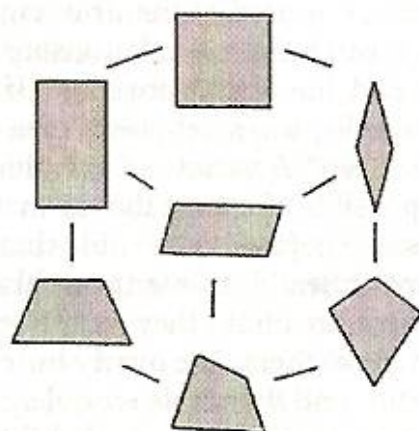


Figure 6

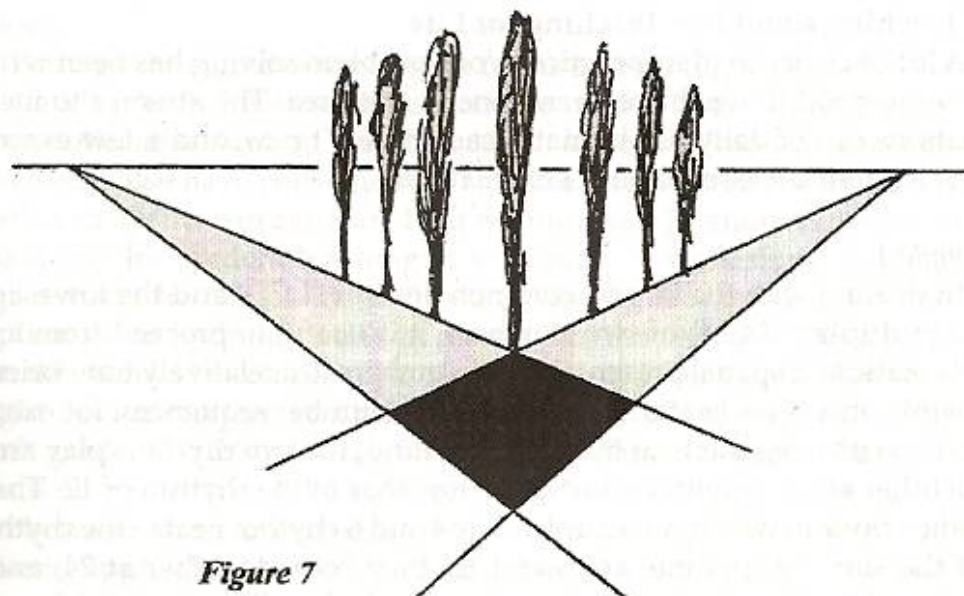


Figure 7

We can give the lessons dealing with this a special turn if the children learn that each single quadrangle has its own position in relation to the square, rather than learning a number of quadrangles such as rhombus, isosceles trapezoid, parallelogram without any correlation to each other. One could say: the children understand the quadrangle in its different modifications. If we work on this in the 6th grade, we can come to the following results: We can point out that the general quadrangle always is a hidden square. A small

drawing in perspective makes this believable without much explanations and awakens curiosity for the upper classes in which what we mean here will be worked on more exactly. With examples like this we can try to motivate the children to think in totalities. Knowledge which is made up of disconnected single facts is more or less unfruitful and is not conducive to creative flexibility in thinking. The tendency to stop at abstract very general concepts will be reduced if a general concept is carried into different individual situations. A lot is said about generalizations and abstractions in math classes. But in addition to these, the ability to think concretely and individually is extremely important for the life of any society. Doesn't the power of a concept really prove itself when it helps us understand the connections between single phenomena and when we can understand connections to other phenomena through the use of it? And don't we suffer from the abstractions of general concepts which are used at the drawing board to describe and govern our lives, even though no real relationship can be established between the abstractions and the actual situation? If we are dealing with a single phenomena or a single process, shouldn't we ask: What is the context in which this should be seen? A variety of conditions and surveys of the situation in which the possible effects of this or that are thought out to the extreme is usually necessary before any sensible changes can be undertaken. Life is new every day, and inflexible thinking can block even the most necessary reforms. When people are afraid they won't be able to understand the changes, they just don't make them. The overly bureaucratic administration of colleges and universities and the whole school system in Germany sometimes appears to be the expression of a lack of ability to think holistically.

All Teaching should be Teaching for Life

A lot of criticism of impractical word problem solving has been written, but some good things have been done in this area. The attempt to include situations out of daily life in math teaching isn't new, and a few examples taken from real classes should suffice here.

Example 1

In dealing with the largest common divisor (LCD) and the lowest common multiplier (LCM) of two numbers, it is useful to proceed from a relatively static and spatial presentation of numbers to a relatively time-oriented, dynamic one. If we beat the rhythm of two number sequences, for example 4 and 6, with a regular beat in the background, the two rhythms play around each other at the beginning but come together in the rhythm of 12. There is another rhythm which can count all the 4 and 6 rhythm beats - the rhythm of 2. If the starting rhythms are 8 and 12, they come together at 24, and the rhythm of 4 is the slowest rhythm which includes all the beats of 8 and 12.

If a and b are the numbers of the starting rhythms, the formula is:

$$a \times b = \text{LCD}(a,b) \times \text{LCM}(a,b).$$

In words: The product of two numbers is the same as the product of their lowest common divisor and their smallest common multiplier (see, for

example, Locher-Ernst 1970). Children can discover this law empirically quite early, and if it is properly handled it can bring about astonishment and awe.

As delightful as this "poetry of number rhythms" is for every mathematician (and where are they nicer than in number theory!), at school it's necessary to put them to use, for example in teaching about gears. Why do gears with a heavy load have cog-wheels in which the number of teeth have no common divisor? (The reason is that possible weaknesses in one tooth won't touch one groove until all the other grooves have been passed. The destruction of the gear wheel by this damage can be postponed quite substantially in comparison to wheels with common divisors.)

Example 2

Waldorf schools, just as other schools, sometimes face the problem of having high school students who are tired of school. There are many ways schools can come to terms with young people's desire to get out of school and into real life. One way is through a practicum. For math, the surveying practical has proved to be very successful. If it is possible, as it was a while back in the 10th grade of the Mannheim Waldorf school, to have a request for help from a community which is surveying its surrounding woods after havoc-raising storms, students have the chance to locate and relocate boundary stones and to apply what they have learned in a very practical way. What has been done in class has a correlation to the world outside of school and leads to helping out in a way which society recognizes. The saying "*non scolae sed vitae discimus*" has never impressed young people as an abstract statement unless they have had the chance to experience its truth outside of school.

Perhaps the significance of such forms of learning for society is not immediately visible because the effects the teacher sees on the students are more noticeable. And on the other hand, lessons without a direct connection to life also have consequences which are important for society. We can see this easily in an exaggerated but not unusual situation. When the author was doing his student teaching at a German "Gymnasium" (college prep high school) he taught the hydrostatic laws and wanted to go into their application in hydraulics and in caterpillars the next day. His advisor told him this would be completely out of place at a college prep high school – it might be appropriate at a school for children not going farther than the ninth grade. The advisor was, otherwise, a valued colleague, but this situation is characteristic of the spirit of a lot of teaching. Isn't there quite a conflict between those in our society who learn about physical laws, but never have to apply them to dirty, oily machines, and those who by and large have no access to this general knowledge, but who work in the world of machines? A different style of thinking in our schools could do quite a bit to further understanding between people working in different positions. We should keep in mind, that a lack of understanding among people who are dependent on each other must cause and has caused damage.

Demythologizing

A demythologized world, in which non-human, spiritual activity has been closed out of modern consciousness, must now face a new mytholo-

gizing of the machine. The computer as almighty Big Brother, who helps when human knowledge has come to an end, is not only popular because of TV or films. Myth building finds nourishment in places where we would never expect it, such as in university engineering or where important economic decisions are being made. See S. Turkle (1984) or W. Volpert (1985). In the discussions among Waldorf teachers a consensus has been reached, that practicing programming etc. is not enough to demythologize the machine. Therefore, contrary to the trends the high school classes at Waldorf schools put their main emphasis on understanding the hardware. Starting with simple switches, which have been self made, classes can work up to understanding the principles of data-processing step by step. Programming can then be added, and other subjects and non-numerical tasks should round the topic off. (Brater/Herz 1986, Schubert 1984/85, Schubert 1988, Schubert 1990).

The student should be able to see quite clearly in which category a problem must be described so that it can be worked through by machine. As described by Schubert (1984/85, 1987 and 1988), a formal system cannot set the boundaries of its validity in relation to a real problem by itself. Human thinking must remain capable of judgment about reality. How should education react to this challenge to human judgment if we don't want to live in an artificial world, as many people living in the half-world of the TV screen do, as responsible members of society? Here are problems which everyone responsible in education should discuss intensively and try to solve.

With these as with all other examples, we are looking for the possible connection between mathematical thinking and social judgment. If we study ways of thinking which are used in society's discussions, or which show up unconsciously in behavior, the connection to mathematics classes and their possibilities as shown here is obvious. The basic question must be: Can practicing flexible and holistic thinking in math have consequences for the life of our society? And if we can answer this with "Yes", which conditions must be fulfilled for a favorable transfer? Can interdisciplinary classes help?

Here the emphasis has been on the development of thinking, but that is only one possible aspect of educating social concern. As mentioned at the beginning, many other factors, which also have to be taken into account in mathematics, might also play an important role. Certain topics can be socially relevant. Teaching methods themselves are part of social education. Class goals, the class atmosphere, the relationship between teachers and students, the acceptance of each individual, the way mistakes are dealt with, the encouragement of group work and many other points would have to be included if the description were to be complete.

Since school doesn't become socially relevant until years later – when the former student has taken over a responsible position – we should lay the ground for these forms of thinking very conscientiously. And to do that, we need the ability to recognize those thinking forms which carry a potential for society in them.

Bibliography:

- Adams, G.: Strahlende Weltgestaltung. Dornach, 1965.
- Adams, G./Whicher, O.: The Plant in Space and Counterspace. Stuttgart, 1979.
- Brater, M./Herz, G.: Persönlichkeitsbildung und Technologiebewältigung. Zu den Aufgaben der Berufsvorbereitung im Computerzeitalter. In: Erziehungskunst 7/1986.
- Buck, P./Mackensen, M. v.: Naturphänomene erlebend verstehen. Köln, 1988.
- Finsler, P.: Aufsätze zur Mengenlehre. Darmstadt, 1975.
- Fischer, R.: Mathematik – Zwischenwelt in Maschinen, Bildern und Symbolen. Ausarbeitung eines Vortrages an der Volkshochschule Bregenz, 1984.
- Franz, M.-L. v.: Zahl und Zeit. Stuttgart 1970; insbesondere 5. 60ff.
- Gschwind, P.: Der lineare Komplex, eine berimaginäre Zahl. Dornach, 1977.
- Gschwind, P.: Methodische Grundlagen zu einer Projektiven Quantenphysik. Dornach, 1979.
- Gschwind, P.: Raum, Zeit, Geschwindigkeit. Dornach, 1986.
- Heintel, P.: Politische Bildung als Prinzip aller Bildung. Wien/München 1977.
- Jungwirth, H.: Sozialisation und Mathematik. In: Endbericht zum Projekt "Mathematik und politische Bildung", vorgelegt von: R. Fischer u. a. Institut für Mathematik, Universität Klagenfurt, 1986.
- Locher-Ernst, L.: Raum und Gegenraum. Dornach, 1970.
- Locher-Ernst, L.: Arithmetik und Algebra. Dornach, 1984.
- Maturana, H./Varela, F.: Der Baum der Erkenntnis. Bern, München, Wien 1987.
- Schuberth, E.: Zum Informatikunterricht an Waldorfschulen. In: Erziehungskunst 6-12/1984 und 1, 3-6/1985.
- Schuberth, E.: Menschliche Kreativität und Künstliche Intelligenz. In: Enquete-Kommission: "Einschätzung und Bewertung von Technikfolgen; Gestaltung von Rahmenbedingungen und der technischen Entwicklung". Materialien zu Drucksache 10/6801, Bd. II. Bonn, 1987.
- Schuberth, E.: Datentechnik, Denken und Wirklichkeit. In: Gergely, E./Goldmann, H. (Hrsg.): Mensch - Computer - Erziehung. Wien, Köln, Graz 1988.
- Schuberth, E.: Erziehung in einer Computergesellschaft. Datentechnik und die werdende Intelligenz des Menschen. Stuttgart, 1990.
- Steiner, R.: Grundlinien einer Erkenntnistheorie der Goetheschen Weltanschauung. GA 2. Dornach, 1979.
- Steiner, R.: Erziehungskunst, Methodisch-Didaktisches. GA 294, Tb 618. Vortrag vom 21.8. 1919. Dornach, 1981.
- Turkle, Sh.: Die Wunschmaschine. Vom Entstehen der Computerkultur. Reinbek, 1984.
- Volpert, W.: Zauberlehrlinge. Die gefährliche Liebe zum Computer. Weinheim und Basel, 1985.

A Vision for Waldorf Education

- A Personal Appreciation of the Works of Bindel
and von Baravalle -
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In Munich, Germany, where I was attempting to do graduate work in musicology, the Waldorf School is out of sight to the casual passersby—only cutoff corners on the McDonald's golden arches and other neighboring store-signs offer clues to the savvy ensconcement of the school in a central courtyard, surrounded by bill-paying rental properties along the street-front of its trendy location on the Leopoldstrasse of Schwabing, Munich's Latin quarter. One of those stores was a book shop, and it was there I first met Anthroposophy in print in the intriguingly illustrated art- and science- related publications of the (as I later learned) two pioneering mathematics teachers from the original Stuttgart Waldorf school, hired personally by Rudolf Steiner—Ernst Bindel and Herman von Baravalle. It was my privilege to meet both of these gentlemen in their last years, as I underwent my own transformation to become a mathematician.

I was working as an orchestra musician at the time, playing bassoon in the Munich Symphony (mostly radio, TV, and film soundtrack work), and enjoying the special sensitivity to ensemble work I had grown up with in Summit, N.J., playing with Bell Lab musicians, found again in various Munich chamber groups associated with the Siemens Corporation, Max Planck Institute, and (for a few lucky weeks) at the home of physicist Werner Heisenberg, who kept a harpsichord and a grand piano in his living room, and practiced some Bach and Mozart daily during mid-afternoon coffee breaks from his laboratory in Freimann.

My first attraction was, accordingly, to the music-related books of Ernst Bindel: a 3-volume work on the historical development of tuning practices, a study of key relationships in the music of Richard Wagner, and a booklet on the role of logarithms in sense perception, plus some profusely illustrated monographs on conic sections and geometrical tilings, culminating in a series of titles on the history of perception of order in nature from ancient Egypt through the work of astronomer Johannes Kepler, which was to grow to become my own central concern. (I now teach a history-through-math course for liberal arts credit at CSUN, attempting to make it a Waldorf main lesson for adults.)

The assorted geometric and arithmetic booklets by von Baravalle seemed less ambitious to me at the time, directed toward teaching "lower" math, not studying "higher," so I passed them by at first. Then I dreamt one night of Alice trying in vain to catch up with the fleeting Red Queen and being told by the live flowers to go the other way, woke up, and realized that my own attempts to pursue "higher" studies were turning out to be just as vain, and that I would do well to teach "lower" instead. So I applied to the Teachers' Seminar at the Stuttgart school and underwent its marvelously rich 2 1/2 year training program for class teachers, though still thinking of myself as a musician, and applying as music teacher to the only Waldorf school I'd heard of at that point in America, the High Mowing School, in Wilton N.H. Its founder, Beulah Emmet, wrote back that they already had a music teacher, thank-you very much (not so, as it turned out), but needed someone to teach math and chemistry, —and I was to be it! Her letter arrived the same week in which I'd learned that, according to Steiner, tone-, number-, and chemical ethers were essentially the same; I knew that Borodin had been both chemist and musician. so I swallowed hard and accepted.

I arrived at High Mowing in August of 1967, and set about cleaning the geometry room and chem lab, to ready the premises for the return of the students in September. There had evidently been a geometry main lesson on the last day of school in June, for the boards had been abandoned, drawings affixed, on one classroom's tables; I separated them, stored the boards, and asked the counselors' help in returning the drawings to individual dorm rooms. The chem lab was full of glassware left covered in black coal tar derivatives by the previous teacher, so I spent the next several weeks cleaning them, thinking of my grandfather who'd been a drugstore proprietor, twirling my towel in flasks as he must have done in soda glasses. Each job drove out thought of the last, and by the time the students arrived I had forgotten August entirely in the press of September.

Mrs. Emmet wisely and kindly gave me the first September main lesson time free so I might observe others before attempting my own first lessons. It fell to me, therefore, to drive to the airport to pick up Dr. von Baravalle, whose 12th grade lesson in physics of optics I was to observe. He identified the Manchester River and its tributaries as we passed over them on successive bridges. Autumn foliage was at its height, so on impulse to enjoy the woodsey scenery, I turned off the road from Amherst to Milford and took an unpaved country road which I judged would roughly parallel the main highway from Milford to Wilton, wondering casually aloud where it might come out. "This would come out at the steel bridge." opined Dr. von Baravalle I didn't even

know what bridge he meant, but sure enough, some 15 minutes later we crossed a small steel bridge and came out at the fruit stand just outside the entrance to Wilton. I was impressed!

The optics lesson went well—it was essentially that which can be found as *Introducing Physics* on pp. 123 ff. of the compendium of lectures transcribed from tape recordings of talks given by Dr. von Baravalle at about the same time in Threefold Farm in Spring Valley now made available in book form from Parker Courtney Press (307 Hungry Hollow Rd., Chestnut Ridge, NY 10977) as *Waldorf Education for America*, painstakingly edited by David Booth and supplied with commentaries on individual topics by, Norman Davidson, Michael D'Aleo, James Henderson, and David Booth. Known principally for his 1957 work on *Geometrie als Sprache der Formen* (never published in an English translation, but available informally in some *samizdat* version at most Waldorf schools), von Baravalle's treatment of optics was, naturally enough, essentially a lesson in applied geometry, but one honed to such a nicety of basics, and presented with such elegant simplicity, that one could only marvel at the wisdom of selection that lay behind its preparation. I was reading a 3-foot shelf of chemistry texts at the time, attempting to winnow them down to basics which I could present in my first lesson (burning autumn leaves, catching smoke going up and ash falling down, and seeing how they changed the color of water boiled with purple cabbage to orange and green respectively), so I knew the problematics of what to include archetypally and what to leave out as details—at least on first pass.

Dr. von Baravalle was noticeably tired, during his High Mowing visit. He sat so deeply in his lounge chair during weekly faculty meetings that one could only guess whether he was asleep or awake. But when time came to be called upon by Mrs. Emmet to say some general words on Waldorf education, up he would lurch to full (again impressive) height and hold forth, always going straight to the point in well-chosen words, and then sit down again, eyes closed, straight again and be gone for all we knew, saving his strength. Several similar talks, intended for parents, may be found in Parts I through III of the collection cited above, dealing with the *Philosophy of Freedom*, the place of a non-"systematic" school in a world of education and the treatment of a child as both an individual and as having typical traits of various temperaments.

As the time drew near for his departure from High Mowing, I requested his advice on what I should best teach in my second main lesson coming up (after the 9th grade leaf chemistry)—it was to be a 10th grade geometric drawing class. He replied laconically—teach

them perspective," and that was the end of the discussion. He'd written a book on the subject, so I duly set about studying that, in preparation. Come the Sunday evening before the first Monday morning of my first geometry lesson, I suddenly remembered all those geometric drawing papers I'd cleaned from the classroom the prior August. They were perspective lessons, and the names on the papers (which had meant nothing to me in August, but which I now recognized individually were those of last year's 9th graders, this year's 10th graders. He had just done with them in June what he had advised me to do with them in October—impossible! What to do, instead?

I recalled the common themes of 10th grade curriculum (Petrarch vs. Shakespeare sonnet schemes in English, balancing acid and base reactions in salt chemistry—seeing how forms arise, and classifying them by types) and decided to conduct experiments to see how regular geometrical forms arise. We dropped rulers from a yard or so height onto our papers, spinning them slightly to obtain random results, and drew extended lines wherever they landed, treating the edges of the paper as four more. Then we colored the resulting meshes by any handy system: red for triangles, blue for quadrilaterals, yellow for 5-sided, and green for 6- or more-sided, tallying the final results to see how they were distributed, on average. As I'd found that fateful Sunday evening, about 40% of the meshes turned out to be 3-sided, 30% 4-sided, 20% 5-sided, and 10% 6- or more. These were then recognized as the point: line *duals* of the ten regular Platonic and semiregular Archimedean tilings of the plane, of which 40% have 3 edges meeting at each corner (such as the hexagon honeycomb), 30% have 4 edges per corner (as in square checkerboard), 20% have 5 per corner, and 10% have 6 (as in triangular tiling). A satisfying order had arisen, unbidden, "naturally," out of seeming chaos! (I have repeated this experiment many times with the same results, only varying by about $\pm 1\%$, but have, alas, to this day no idea of what the ideal theoretical distribution might be—suggestions are welcome!)

I later learned that Dr. von Baravalle had experienced difficulties giving his next main lesson at Highland Hall in Los Angeles, the week after his departure from High Mowing, and had to be relieved of the class. I had been fortunate to experience the delivery of his last perfect lesson (on optics), and also been fortunate to be given a creative miscue on my own further development.

Two and a third years later (New Year's Eve, 1969) High Mowing suffered a severe fire in which all of my possessions were lost (in particular, all of my music). Since I had by that time become a mathematician

de facto thanks largely to the generous and wise guidance of physicist colleague James Huston), it was time to receive some official credential training in mathematics. When Highland Hall requested that I help open their new 11th and 12th grade classes, offering as incentive the proximity of then San Fernando Valley State College (now C.S.U. Northridge) for master classes at night, I accepted.

The next year (1971) Highland Hall offered to pay half my fare to fly to Detroit to experience lectures by Lawrence Edwards from the Edinburgh Steiner School, who was applying projective to modeling plant growth, building on prior work of George Adams who had been entrusted by Elisabeth Vreede to develop what was then called "newer geometry" (i.e. the projective archetype which becomes various Euclidean and non-Euclidean geometries when metrically specialized). This led to my becoming editor (from 1972 to 1982) of an English version of the quarterly *Math-Physics Correspondence* (freely paralleling that edited by Georg Unger in Dornach) to give Edwards a forum in which to present this work. When I went to Seattle in 1974 to complete my master's degree and then on to Edinburgh to take over Edwards' classes and give him a sabbatical to study plants in Australia, David Booth took over my former classes at Highland Hall, on his way to eventually landing in Spring Valley - so our several paths have crossed.

As Booth observes in his introduction to the above collection (P. 3), the 1960's were a time of pedagogical reaction to "the Sputnik flap," leading to "an impulse [in] educational reform which, in mathematics, became identified with the new math. This movement favored abstraction and did not leave room for geometry lessons like those von Baravalle had advocated for the junior high years." Baravalle had spoken at conventions of the National Council of Teachers of Mathematics and contributed numerous articles to leading educational journals, all well received by older teachers—but the times were changing, and geometry was being passed over in favor of set theory even as developmental psychologists such as Piaget were showing that set theory is the *least* effective way to convey mathematics to children. The syllogism ran something as follows: "Set theory is the most primitive mathematics, children are the most primitive human beings; therefore children should first be taught set theory."

I must confess that I was instrumental in beginning the reduction in time allotment to geometry in main lessons at Highland Hall during my stay there in the early 1970's. Dr. von Baravalle had left a tradition of two geometry main lessons in every year, from 9th through 12th grade, but other areas of curricular importance were being neglected. There was, for example,

no geography lesson, and Dr. Steiner had said (somewhere, as I remembered it) that the attention or lack of attention devoted to geography could be used as an indicator of the general humanity of a school. I took this seriously, and shortened one of the 11th grade geometry lessons from the abstract rendering of arbitrary solids (what I had in high school as "mechanical drawing" or "drafting") to allow time during the third week for the special treatment of projections in global map-making.

A number of gray plastic globes were purchased (for half the number of students in class), purposely featureless except for contours of continents, on which half the students, working in pairs, used grease pencils to draw latitude and longitude lines, based on their choice of central and antipodal points (in lieu of usual north and south poles)—the other member of each team prepared two sets of papers, one pair of opaque sheets and one pair of translucent ones, marked with corresponding grid lines, so that each could then draw the front half of the globe on his or her own translucent sheet, and then use the same square-for-square transfer process in mirror reverse to draw the back half of the globe on the opaque sheet, so that when overlaid the result was as though viewing the Earth transparently from front to back, and when done to be labeled with the student's choice of amount and style of geographical identifications. There was some frustration at first in the reversed drawing, especially due to the blankness of the gray globes being copied. One new student complained bitterly that she "didn't even know what [she'd] just drawn." "You just drew Vietnam," I replied quietly in the shocked silence of the classroom. It was the era of greatest antiwar protest, and she had unwittingly made the intended point of the lesson perfectly. There were no further protests—at least not of the drawing lesson.

I have just returned from a conference on "Bridges" between mathematics, art, and music—one of six held internationally this summer in places such as Padua and Tel Aviv. This one was at a small college in rural Kansas, but the attendees were no less remarkable. A U.C. Berkeley computer graphics expert from Basel helped a Missouri farmer realize his intuitive wood sculptures of intricately wound ribbons with negative curvature soap bubble geometry applied to complex saddle surfaces, or a math grad student with a borrowed Guarneri violin performing concertos by request, note-perfect, from memory, during the after hours. (The Proceedings may be ordered from Prof. Reza Sarhangi, Dept. of Math., Southwestern College, Winfield KS 67 156-2499: ph. 1-800-846-1543.)

The *Los Angeles Times* published an essay by K.C. Cole on "Why the Arts Are Important to Science," (see *Waldorf Science Newsletter* Volume 5, #10 for a full reprint of this article) citing the beginning-to-be recognized pedagogical purposes behind the gradual restoration of art and music lessons in our public schools. Writes Cole, "One obvious reason is that scientists, like artists, must learn to pay close attention—both to detail and to the broader context. Scientists, like artists, are people who notice things." That is what I have come to recognize, in retrospect, was the single most valuable (and rare) lesson I received from playing chamber music with personnel from the Bell Labs during my high school years. They listened to one another! In our *Messiah* performances woodwinds, strings, and vocal soloists ornamented in mutual consideration—that does not happen otherwise, among "just" musicians (woodwinds chatter, strings play what's printed on the page, and vocalists know one appoggiatura). But it did among the Bell Lab musicians, it did among the Max Planck Institute researchers, and it did at the home Werner Heisenberg. Whole education trains coordination of eyes, ears, and limbs, responsively—and responsibly.

Our Approach to Math Doesn't Add Up

As the debate rages over American students' math failures, the finger of blame has been pointed at teaching methods and class size. But almost everyone overlooks the students as a solution—more needs to be asked of us.

Take one of the latest dismal headlines on the subject: "Test Shows That Students Can't Do Math; 64 Percent Fail Final Exam After Montgomery Standardizes Grades" [*Metro*, April 30].

A couple of years ago, I took that final when I was an advanced student in the Montgomery County school system. I had won the math-science award, got A's in advanced math courses and was honored with wonderful scholarships. I did well on that final exam. I thought I had learned a lot of math. But I was wrong. I am a sophomore now at Northwestern University. Studying math is different here. Of course, classes are more difficult at college, but the difference is much more basic: I have no guarantee that I'll "get it," and no one pretends otherwise.

Sometimes I can work for hours on a problem and not be able to find the solution. Then I sit longer, forcing my mind to bend in new ways, and the concepts begin to come clear.

The process feels awful. Each time I go through it, I wonder if my brain is flawed, if I have reached my limit. Sometimes, when I finally do get the answer, it is too late for the test. So I have failed some tests. But I keep going because if I don't face the possibility of failure, I have no chance to succeed. Its only in the difficulty that we become able to think mathematically.

No one is telling American high school students about this hard reality. In fact, they are being told otherwise. The Montgomery County motto is "Success for Every Student." Who could argue with that? But if success is guaranteed to every student, if failure is not an option, then hasn't success been redefined to include failure?

When our high school math teacher skipped a major curriculum unit, he hid that shortcoming by showing his class the type of problems that would be on the final and how to answer them, but he didn't explain the content. Flagrant student cheating goes unchallenged as well. When my principal was asked about his failure to address widespread cheating, his defense was that in the real world, people don't always get caught.

With attitudes like that, it is no surprise that high school students are unprepared for the rigors and realities of that real world.

For example, a high school student I know who received A's in precalculus scored so poorly on the University of Maryland placement exam that he had to retake the course in college. Even then, the second time around, he could manage only a D. Students need to be told the truth about math. Math is hard and will challenge us. But American students can rise to challenges. We don't need it to be too easy. We need it to be real.

- Genevieve Maricle - *The Washington Post*,
May 21, 2000

TESTING STUDENTS



The following is taken from *The Educational Tasks and Content of the Steiner Waldorf Curriculum*, edited by Martyn Rawson and Tobias Richter as research produced on behalf of the Pedagogical Section of the School of Spiritual Science at the Goetheanum, Dornach, Switzerland. ISBN 1 900169 07 X published by Steiner Schools Fellowship, 2000. Inclusion does not mean endorsement. In fact, we hope to receive many letters sharing how differently we do it in North America.

– David Mitchell

ARITHMETIC AND MATHEMATICS IN A WALDORF SCHOOL

Classes 1 to 8

General aspects and aims

*"Mathematics in the Waldorf School is divided into three stages. In the first stage, which covers the first five classes, mathematics is developed as an activity intimately connected to the life process of the child, and progresses from the internal towards the external. In the second stage, covering Classes 6 to 8, the main emphasis is on the practical ... The ninth year onwards is characterized by a transfer towards a rational point of view." Thus writes Hermann von Baravalle, the first mathematics teacher at the Waldorf School in Stuttgart, about the treatment of this subject, in his book *Mathematics Teaching and the Waldorf School Plan*.¹*

The first stage

Here two questions must be answered:

1. How should the first mathematical concepts be tackled?
2. What is the psychological basis on which to build them?

Question 1: Careful scrutiny shows that the teaching of arithmetical and geometric concepts is connected to the consciousness and activity of the child's movement organism. Counting is inner movement by which outer movement can be observed. E. Schuberth calls this *"the sensory content of mathematics teaching"*.² The results of Piaget's research on the development of intelligence in children also point in this direction: In the 'concrete operations stage' (12 or 13 years old), children still carry out movements when they want to connect one thing to another. Anyway, these movements are connected to the physical objects from which the children can as yet barely free themselves, if at all.

This leads on to Question 2: If the development of mathematical concepts occur, in the static, concrete phase, our purpose must not be to *'generalize and abstract'*, but to *"make concrete and look at individual cases"*.³ This defines the means whereby it is possible to avoid confronting the children with abstract logical structures, but rather to immerse their whole capacity for experience in mathematics. We can now refer to form drawing where the consciousness necessary for using mathematics is nurtured and practiced. This physical experience is a basis and assumption for a healthy immersion in the 'formal operations stage' (Piaget). The rule 'from hand, through heart to head' (which is meant by the child's above-mentioned 'whole capacity for experience') makes it possible for the children to bring their own capacities into play.

*"It is evident that the best, most fruitful questions about concepts and explanations come from pupils who do not pose questions using a quick intellectuality but a capacity for involved feeling, which allows clarity to come into thinking."*⁴

To this concrete approach in mathematics at primary school level we should add something further which does not depend on **the element of movement**. This is the quality, one might say the identity, of the individual numbers.

As the accent above is on a quantitative approach to numbers, as the result of a brief pause in movement or even based on the movement itself, we need to place our introduction to qualitative number concepts beside those of quantity. We approach these qualities when we examine many examples where the number in question is really active in the world, as for example the number 5 in the flowers of a rose. Here we use the child's desire to question what lies

behind the world and human creations, that is, to seek what lies behind the phenomena. The nuclear scientist W. Heider referred to this when he said in a lecture: *"One directs one's attention to qualitative phenomena, to characteristics which have something to do with the totality of the objects observed"*. Steiner recommended taking this as the starting point for an introduction to number concepts: *"We have gradually come to the point in the course of civilization. Where we can work with numbers in a synthetic manner We have a unity, a second unity, a third unity and we struggle while counting in an additive manner to join the one to the other, so that one lies beside the other when we count. One can become convinced that children do not bring an inner understanding towards this. Primitive human beings did not develop counting in this way. Counting began from unity. Two was not an external repetition of unity but it lay within unity. One gave us two, and two is contained in the one. One, divided, gave us three, and three is contained within one. If we wrote the number 1, translated into modern terms, one could not get away from the 1, for instance to 2. It was an inner organic picture where two came out of unity and this two was contained in the one, likewise three and so on. Unity encompassed everything and the numbers were organic divisions of unity"*.⁵

This 'real' way of looking at numbers leads on to written numbers, to symbols. It is not a picture like that which should be used for introducing letters, but pictures of number qualities. The picture belongs to the being of the number, not to the outer symbolic form. At this point we indicate a further point of this 'quality-oriented' teaching: Today, especially, when we face the results of a quantitative world view in ecological catastrophe and destruction, it is increasingly important to make a beginning of this kind in teaching mathematics.

By beginning with the concrete qualities of number and by working with the properties of movement in counting and calculating, children develop a kind of intelligence which seeks and finds the way to reality.

This brings us to the *second stage*: the approach to mathematics teaching mentioned earlier. Here we must deal with the practical use of calculation.

If calculating has been practiced thoroughly enough during the first stage in the manner indicated, then applied calculation also gets a qualitative coloring. The forces of intelligence which are served by business mathematics as well as percentages and interest, are not 'value-free', but can retain a coloring for balanced testing and judging. The human significance of what is thought out, can and should be made clear. In this connection one should point to the suggestion by Steiner that elements of bookkeeping should also be taken up in mathematics lessons. To see what the general idea behind such an indication is, one should ask what skills may be developed through bookkeeping. There you will see how above all a moral method of trading can be decisively supported by these means.

All these themes can lead us on to other educational aims: inner mobility leads to imaginative ability in solving mathematical problems.

Through the experience of number qualities the children experience trust and security. Number, world and human being belong together.

The children can experience further security through the correctness of solutions to problems. By this means they win some independence. *"I or this reason mathematics is an activity suited to free children from the fetters of authority even if, at first, they depend on the help of the teacher"*⁶

A final educational aim which should not be undervalued and which is connected with the latter: calculation is not possible without regular practice, which also makes it an excellent medium for schooling the will.

An explicit presentation of the *third stage* will be given under the heading 'General aspects and aims. Classes 9 to 12 and is therefore omitted here

Geometry as part of mathematics teaching begins in Classes 5 and 6 and is taught in separate main lessons. One of the principal intentions in this subject is to develop and nurture the ability to visualize space.

The controlled security of directed movement and the estimation of proportions and relationships, practiced in freehand geometry, is well prepared for by form drawing in Classes I to 4.

The establishment of skills, knowledge and techniques partly in connection with subjects is taught with increasing complexity related to age

- The pupil should gradually learn to discover, to grasp mentally and to apply geometrical properties, and the practical, drawn, solutions representing them.
- Work with geometric drawing instrument, should lead to clear and exact construction.
- Patience, care and precision should be developed, as well as independent creative work through enjoyment of drawing

Classes 1 to 3

Points of view and general themes:

The dynamics of will activity should be internalized by the experience of countability. Motivation should be awakened through pictorial description of number qualities. This dual aspect is important: on the one hand it educates the bodily senses through experience of movement, unfolding of movement possibilities (both coarse and fine), and co-ordination exercises. On the other hand internalizing of the expressed activities in soul activity (i.e. calculation). Here

the main medium for achieving this is the use of pictures. Through pictures children can grasp internally what is intended. Pure symbolic, logical presentation can never achieve this. (Nevertheless one should always be conscious that calculation is aiming at a pictureless world, in contrast to the introduction of letters of the alphabet.) In order to be able to handle quantitative numbers freely, an inner numerical space needs to be created, in which one learns to move, rhythmically at first, with varied number patterns. This is achieved, amongst other means, by a memory developed by learning the times-tables through rhythmic movement e.g. through clapping, passing bean bags or skipping. It appears important initially to approach actual calculation as concretely and visually as possible and to keep in mind the principle 'from the whole to the parts'. This means that the right connection between analytical and synthetic thinking is produced. Work with the temperaments should be formed in the sense given in the 4th discussion in *Discussions with Teachers*.⁷ At the end of Class 3 the pupils should have a confident grasp and clear view of numbers up to 1020 at least. This does not mean just the quantity or extent but equally the quality of the numbers.

Content suggestions:

Class 1

In mathematics lessons the approach is analytical in the sense of the reference by Steiner given above. Starting with the number I as unity, numbers (symbols) from I to 10 should be produced in a qualitative manner (see 'Points of view and general themes' above), which are contained as manifold in unity. With written numbers one can begin with Roman numerals which are less abstract than the Arabic.⁸ Alternatively, the Arabic numbers can be introduced in pictures as with the letters of the alphabet.

- Counting up to 110
- Learning up to the 7 times table by heart and through rhythmical practice
- Introducing the four rules using numbers up to 20 and also in written form (in notation the sum is written first
7 is $3 + 4$)
- Number riddles
- First exercises in mental arithmetic

Class 2

- Further practice in mental arithmetic
- Extension of counting and practice of the four rules using numbers up to 100 Practice in combined calculation
- Initial consideration of number connections 'Kingly numbers and 'beggar' numbers (primes)
- Up to the 12 times table by heart
- Representation of tables in drawing
- Written analytically and synthetically practiced calculations
- In the course of Class 2 calculations should be reversed (e.g. $3 + 4 = 7$)

Class 3

- Mental arithmetic
- Sums using numbers up to 1020 or 1100
- Written addition and subtraction using several places (place value)
- Written multiplication using two place values
- Written division using units as divisor
- Up to 15 times tables 10 times table up to 900
- Square numbers by heart as a sequence
- Weights and measures (practical subject) and calculations with simple practical problems

Checklist for Classes 1 - 3 - Numeracy

NUMBER

Most children within the normal range of ability will be able to:

Grade Topic(s)/Skills

- 1 have working knowledge of four processes and their symbols $+$ $-$ $\times$ $\div$ (including processes in verbal and written sentence form)
- 1 appreciate number qualities 1- 12
- 1 understand Roman numerals I - X and Arabic numerals 1-100

- 1 count from 1-100
- 1 know number bonds up to 10
- 1/2 understand difference between odd and even numbers
- 1-3 have work knowledge of the multiplication tables 1-12
- 1/3 apply simple mental arithmetic in narrative form using above listed skills
- 2 know number bonds up to 20
- 2/3 recognize, analyze and count to numbers up to 1,000
- 2/3 work with tables as division (24 shared between 6 is 4)
- 2/3 know patterns in multiplication tables 10, 9, 5, 4, 11
- 2/3 use place value to four places (ten, H, T and U)
- 2/3 carry numbers across columns e.g.

$$\begin{array}{r} 19 \quad 74 \\ +2 \quad \underline{x2} \end{array}$$

- 3 be able to recite tables 1 - 12 in chorus and individually

FORM DRAWING (see form drawing curriculum)

- 1/2 draw straight line, curves, linear forms, symmetry on vertical axis
- 1-3 draw common geometric forms freehand
- 3/4 draw symmetrical reflections: about horizontal and diagonal axis

MEASUREMENT

- 2/3 use money for simple bills and calculating change
- 3 tell time using hours, half hours, quarter hours on 12 hour clock
- 3 calculate simple practical sums e.g. how many milk bottles in a crate holding six by six, bricks in a wall, floor boards etc.
- 3 calculate simple sums in measurement of length, capacity and weight

Classes 4 and 5

Points of view and general themes:

Upon reaching their 9th year the children make a decisive change. Their close relationship to the world around them becomes different and more remote. The earlier harmony between outer and inner worlds is fundamentally broken.

This transformation in their soul is reflected in the mathematics curriculum when in Class 4 the children begin to work with broken numbers (fractions). By this means they meet something in the teaching content which they have also experienced in themselves.

It is not essential for the children to be able to manage fractions swiftly. It is much more important that they can experience these 'external' fractions very strongly. In connection with this, the historical development of fraction calculations in Egypt can give the teacher interesting and significant teaching ideas. In order to do general justice to the subject of fractions it is recommended to use the following three methods as an introduction: To proceed from the whole to the parts, from the parts to the whole, and to establish the principle of equivalence. After this the four rules are practiced with fractions, the same with simplifying, expansion and division of the denominator into prime factors.

After this, decimal fractions follow as a practical application. Once the divisibility border is crossed the children can discover the practicality of calculations in Class 5.

The aim according to Steiner is as follows: In Class 5 we want to continue with fractions and decimals and to give the children everything which will allow them to calculate freely with whole and fractional numbers.⁹

In Class 4 form drawing is led into elementary geometry. Here one can begin again with the basic linear polarity of circle and straight line. In order that the pupils get as intensive an image as possible of these forms, it is recommended that they do not initially use compasses and ruler, but draw free-hand.

Although we deal with the most basic elements in the first geometry lessons, it is important to let the pupil feel something of that dimension which is connected with existential questions over and above the practical and utilitarian. This is made far easier if the beauty of form and strongly regulated connections of geometry are felt in addition to the working rules and methods.

In connection with stories from ancient Egypt in the history lessons, the Pythagorean rope can be introduced as a first introduction to Pythagoras' Theorem.

Content suggestions:

Class 4

- **Mental arithmetic**
- **Practice of written calculations** using higher numbers
- Introduction to fractions: experience of a fraction as part of a whole. From part to whole, similar fractions and different fractions. Transfer of improper fractions into mixed numbers and the reverse.
- Introduction to decimal fractions
- Revision: the four rules and written multiplication and division with several place values
- Freehand form drawing leads into geometrical drawing. Circle, square, triangle, isosceles and right angled triangle. Division of circle into 4, 5, 6, 8, 12 equal parts. This is done by organized guess work. Coloring and imagination will bring out different patterns

Class 5

- Constant practice in mental arithmetic
- Revision: The four rules with natural numbers
- Combinations of the four rules
- Calculations with fractions: Expansion and reduction of equivalents (division into prime factors)
- Illustration and comparison of fractions. Calculation with decimals. Consolidation of fractions methods
- Table of place values, rhythmically, through movement, and qualitatively introduced
- Introduction of the relationship of decimals to place values
- Measurements using decimals
- Recognition of connections between decimal numbers and decimal fractions

The main new task for Class 5 is learning to use a pair of compasses with accuracy, though some teachers prefer to wait until the beginning of Class 6. The forms previously drawn in Class 4 can now be accurately constructed. Children will naturally color these flower-like forms, and thus make an obvious link with the botany main-lesson in Class 5. A set square and ruler can also be used to draw accurate parallel lines.

- Starting with the construction of a circle, discovery of the main geometrical figures: triangle, hexagon, square, rhombus, parallelogram, octagon
- Division and joints on 24 point circle
- Construction of perpendicular bisector, angle bisection, perpendicular bisector, angle bisection, perpendiculars
- Construction of different triangles; equilateral, isosceles, scalene, right angled
- The various angles; acute, obtuse, reflex
- Circles touching a triangle; inside (incircle) and outside (circumcircle)
- Pythagoras' Theorem; visually using knotted string. (Egyptians used this to construct their pyramids). Grains covering an area, theorem drawn using Roman tiles (Isosceles triangle)
- Tessellation (tiling) involving accurate construction of parallel lines
- Exact construction of pentagon/pentagram

Checklist for Class 4 - 5 - Numeracy

Most children within the normal range of ability will be able to:

NUMBER

- 4 carry out all four processes of number confidently
- 4 read and understand numbers up to six figures know the multiplication tables up to 12 out of sequence
- 4 do long multiplication with numbers up to 122 as multiplier
- 4 find factors of a given number
- 4 identify prime numbers less than 100
- 4/5 answer more complex mental arithmetic questions involving a mix of processes (e.g. The 12.38 train to Reading takes 18 minutes but left 14 minutes late, when did it arrive? or I doubled a number and added 8 and got 32, what was the number?)
- 4/5 do long division including making use of remainder and estimating approximate answers.

- 4/5 find Lowest Common Multiple or Highest Common Factors
- 5 use all four processes with fractions including mixed numbers and improper fractions
- 5 understand how to use decimal notation, decimal fractions and interchange of decimal with common fractions
- 5 carry out four processes with decimals
- 5 use long division and multiplication using the decimal point
- 5 apply the Rule of Three (if, then, therefore) to practical problems

MEASUREMENT

- 4 record information such as height, weight, volume etc.
- 5 work with metric measurement including estimation
- 5 work with aspects of time including 24 hour clock
- 5/6 calculate average speeds

GEOMETRY

- 5 draw freehand archetypal geometric shapes: different kinds of triangle, rectangle, quadrilaterals, polygons and circles
- 5 divide circles into 17, 16 or 20 parts, deriving regular figures like pentagon and hexagon from them 10

Classes 6 to 8

Points of view and general themes:

So far, concept-building about method given pictorially has been rooted in an approach to the child's soul. Now, after the 12th year, children can increasingly create order out of what has been gained with the strength of their ability to experience internal logic. This step is exemplified in *algebra*: It leads from the activity of calculating to observation of the processes and from there to the discovery of general relationships.

The purpose of an algebraic formula, of 'calculating with letters of the alphabet' is to express the formal, intelligible processes. This is a general step forward in the development of the child as only the method is formulated: By this means the transfer from an imagination-bound thinking to a conceptual thinking is facilitated ... The process, the delineation of a concrete problem (interest), the solution of the problem, the evidence of the validity of the solution method, and finally the applicability of the discovered rule. All this would be experienced by the children in many situations. ¹¹

As the children approach puberty their feeling life expands in all ways. Mathematics can offer an important support in this stage of life. Their own subjective opinions and ideas are not required! Mathematics attracts their attention not only to the numerical material but especially to their own thinking. If the pupils manage to become confident and secure with mathematical laws, they learn self-confidence. When this is achieved the young people are on the way to the most important aim in mathematics teaching: that of gaining trust in thinking.

However, this thinking can now connect itself in a one-sided selfish way to its mentor, the human ego, but this leads to egoism. It is essential to link thinking to world interests in practical and necessary life situations. It is, however, important that the attempts to solve problems do not lead to resignation with the 'I can't do that' attitude, because mathematics lessons then achieve exactly what they should not. Instead of enjoyment and confidence, they create boredom and despair. There is hardly any other subject which is so equated to scholarly ability and intelligence as mathematics. To 'give up' here or to have problems means to give up generally, and simply to be 'stupid'.

For this reason, mixed ability classes make particular demands on the teacher as regards method or possibly even remedial measures. During the class teacher stage, what the pupils have to do must be differentiated although all of them must deal with the basic mathematical questions. Work with practical problems offers a rich fund of activities for the pupils and can even be formed into a life skill, which might open various avenues to the real world of work. Working by means of mathematical exercises to make thinking energetic fosters an active connection to these areas. The practical activities bring the pupils towards life and reality and also to a description of basic connections.

Calculation is an education of the will in the area of thinking. For this reason practice lessons are added to the main-lessons from about Class 6 onwards.

The precision and beauty of geometrical figures are the teachers who will lead them to greater awareness. What has been experienced through amazement in geometry in Class 5 should be worked on in thinking in Classes 6, 7 and 8. Geometrical rules are sought and formulated. The pupils must also experience geometrical proofs adequately. It is important for them as they develop their individual forms of speech and expression that they can experience something like this, which is quite free of emotion and concerns itself purely with what ought to be. In Class 8 one can use the new subject of conic sections to approach the problem of infinity as one did before with parallels. Infinity is still not defined specifically.

Content suggestions:

Class 6

- Continuing with mental arithmetic
- Revision: calculation with natural numbers, positive fractions and decimals
- Unitary method, with direct and inverse proportion
- Percentages
- Application of percentages to business: interest, discount, exchange, profit and loss, VAT, general introduction to the use of formulae by means of simple interest
- Block graphs and pictograms

GEOMETRY

- Geometrical proof of sums of angles of triangle: using cut outs, protractors
- Proof of above using calculations
- Accurate construction of angles using compasses, bisecting angles
- Construction of triangles from description
- Congruent triangles; the four principle cases for congruency
- Translations; movement properties of triangles and quadrilaterals; crown transformations, triangles in the same segment of a circle (coloring enhances this.) Thales' Theorem
- Leaf forms from; triangles, circle.
- Caustic curves, envelopes of a cardioid
- Congruent shapes, construction of similar angles: Complementary, supplementary and other angles
- Construction of triangles, with altitudes, and angle and side bisectors

Class 7

- Continuing practice in mental arithmetic
- Revision: the four rules in natural and positive rational numbers
- Basic bookkeeping
- Introduction to negative integers (through debt calculation)
- The four rules with negative numbers
- Extension to cover all rational numbers
- The four rules with rationals and their connections Introduction of brackets
- Recurring decimals, later on the value of pi Full understanding and comparison of decimal places and significant figures
- Compound interest
- Simple statistical data rendered in graphical form and deductions therefrom

ALGEBRA

- Simple equations, including brackets, fractions and negative numbers. Their practical application to solving problems
- Making and transforming formulae
- Powers and roots of numbers. The exact evaluation of square roots
- Ratio and proportion
- Calculation of the areas of figures bounded by straight lines and circular arcs
- Types of quadrilateral and their symmetries, leading to simple set theory

GEOMETRY

- Areas of geometrical shapes through construction and calculation
- Area of circle, and using this to calculate the value of pi, by cutting the circle into pieces
- Pythagoras, theorem; area proof
- Shapes and stretches of simple shapes
- Tangents to circles
- Further transformations of pentagons. Construction of decagon and polygons
- Perspective drawing. (Can be linked with modern history main-lesson)

REVISION

- Fractions
- Squares and roots
- Equations
- Practical problems

ALGEBRA

- The commutative, associative and distributive laws in algebra. The factors of the difference between the squares and the application of this to practical problems
- Volumes of rectangular blocks, pyramids, prisms, cylinders and cones. Density and weight of solid objects
- Simultaneous linear equations and problems
- The dissolution of complex brackets in algebraic expressions
- A brief look at balance sheets and mortgages
- Number systems. Binary arithmetic
- Further statistical work including mean, mode and median
- Graphs of more complicated curves. The solution of simultaneous equations by graphs

GEOMETRY

- Locus of line and plane
- Locus and conics defined geometrically
- Enlargements, rotation, reflection of shapes
- Angle properties of circle (angles in the same circle, intersecting chords)
- Construction of 5 regular Platonic solids. Orthogonal view of them
- Exact spatial perspective drawing including the golden section
- Discussion of general triangle sides and altitude formulae as part of the development of the investigation of Pythagoras' Theorem
- Optional: Internal and external angles of a polygon
- Similar figures especially triangles

Checklist for Classes 6 - 8 - Numeracy

Most children within the normal range of ability will be able to:

NUMBER

- 6 convert percentages to fractions and vice versa
- 6 estimate results by rounding off number prior to accurate calculation
- 6 business math: balance sheets: profit and loss, discount, commission, VAT and book-keeping, bank accounts
- 6 work out averages including speed
- 6 read co-ordinates (e.g. for map reading)
- 6 use letters in formula
- 7 know powers of numbers
- 7 work out ratio and scale
- 7 use algebra as a general solution to specific problems
- 7 use negative and positive integers
- 7/8 know how to work with square roots
- 7/8 calculate compound interest, mortgage rates, income tax
- 6/7 make time and speed calculations
- 7/8 calculate mechanical advantage in simple machines e.g. pulleys, levers

DATA

- 6 present information via pictograms: use pie charts, bar charts, linear graphs (foreign currency exchange)
- 7 use algebraic graphs

GEOMETRY

- 6 make precise use of compasses, ruler, set squares to draw constructions of major geometric figures
- 6 make use of freehand perspective
- 6/7 use protractor
- 6/7 draw translations, reflections, rotations
- 6/7 know Pythagoras Theorem and its applications
- 7 use instruments to draw linear perspective
- 7 know properties of triangles, parallel lines and intersecting lines
- 7 know and apply formulae for area of regular geometric forms, including triangle, circle, parallelogram, derivation and use of
- 7/8 calculate areas of irregular forms

Classes 9 to 12

General aspects and aims:

The central point of mathematical activity is problem solving. The important question is how to solve problems, not what one gets as a answer. With this as the focus, school mathematics builds on both bases of mathematics: inspiration (induction) as a beginning and logical conclusion (deduction) at a later stage in the mathematical activity.

The most important aim will be to develop the ability of the pupils to think with a wide range of approaches until they get to the logical conclusion, and to give them confidence in themselves and in their thinking.

Another justified aim is to prepare the pupils to apply calculation methods to everyday life and also to give them the foundation for further education.

It is nevertheless valuable for the principal aim to give problems in many new contexts: It is more important than the division into subject areas, such as algebra, functions etc., to pose questions in various heuristic ways in order to 'unravel' a problem.

The pupils may rehearse their ability to guess, to test variations in an investigation and practice making theories. In order to find a clue for the solution one can simplify the problem; likewise, it helps to make analogies or to generalize the question in order to get an idea of the recurrent theme.

If there is a large proportion of creative problem solving, mathematics can have enormous value for the pupils' development in this period. They are given the opportunity to observe their thinking in a variety of ways: seeking points of departure, choosing examples or counter examples, systematically running an investigation and proving the results. They learn to analyze and to judge conditions.

The artistic educational quality of Steiner Waldorf education is especially evident in mathematics. One can create a fruitful dialogue with the pupils as well as between them. The curriculum indicates the aims and teaching content and curriculum questions, but how the lessons are formed is independent of the required knowledge of the pupils. In this regard it is an advantage if the teacher gradually forms a collection of problems tailored to a particular class.

It is important that pupils are allowed to make inner conquests of general value. They get most value from results which they guessed at first and were then able to prove.

As thinking is an expression of our ego activity, mathematics affords quite special opportunities for the pupils' inner development and self-knowledge

In *geometry*, which can be taught as a separate subject within the framework of mathematics, the following applies in these classes:

- The pupils should practice methods of construction in 3D space.
- The pupils should learn to think in processes; they should break through to, and release, habits of thinking and habits of understanding, and thereby bring more openness and mobility into their thinking.
- The representation of spatial reality such as orthographic and other projections and perspective, are practiced and investigated for their applications.

Class 9

Points of view and general themes:

With the introduction of combinations and especially when beginning with probability - subjects with which Class 9 often begins - the pupils can experience that the thinking which can reach out over what is given and can master these general cases offers them a practical tool for formal logical thinking so that even without special conditions prepared in the Lower School they can effect a 'new entry'. The study of equations, which is extended and deepened, offers a good field of work for their growing formal abilities through its transparent methods of solution. In addition, there are all kinds of series which lead the pupils into deeper study.

A further study of triangles with this new approach offers a medium for experience in simple proofs, whereby things already learnt already are brought into use (e.g. congruence rules from Class 8). The method is analytical, growing from the concrete to the general, from the geometrical construction to the proof. In geometry, work on the conic sections, which may be begun earlier and can be extended later on, offers a variety of construction methods, to create mobile ideas, which are also worked out by rigorous method. In the construction of one of the principal curves (ellipse, hyperbola, parabola), the subject of infinity which has been latent since Class 6, reappears more clearly. At the same time exercises should lead the pupils to a clear experience of the 3 dimensions. One can begin with the cube which visibly represents dimensionality. From this the most varied solids can be developed. Exercises are given to transform these in stages so that the pupils learn to picture them in a fluid way.

Oblique plan views are used to represent these.

The pupils should get to know the personalities whose thoughts they are using, through biographies or mathematicians (e.g. Pascal, Fermat).

Periodic excursions into irrationals and incommensurables with their polarities prepare for the unification of arithmetic and geometry in analytical geometry in Class 11.

At the appropriate moment the calculator can also be introduced.

The wealth of possible content does not permit a comprehensive tackling of subjects. The important thing for Class 9 is the 'how'. It depends on bringing to life concrete representative examples of general rules.

Content suggestions:

KNOWLEDGE AND SKILLS IN ELEMENTARY ALGEBRA

Revision of:

- Natural numbers, integers and rationals Divisibility rules, HCF and LCM
- Primes and the question of how many there are
- Calculation with four rules of polynomials and fractions
- Squares and roots
- Irrationals and the new subject of the domain of real numbers
- Direct and indirect proportion in connection with practical life (percentages and interest)

Algebra

- Linear equations in 2 and 3 unknowns
- Depending on the class ability: quadratic equations (can be done in Class 10)

Combinations:

- Permutations
- Variation
- Optional: basic elements of probability stemming from questions using combinations
- Elements of number theory (bases, particularly base 2 in consideration of computers)

Binomial theorem:

- Binomial coefficients Pascal's triangle
- Calculation with squares and roots, a look at the cube root
- Tricks for simplifying calculation without electronic aids based on the binomial theorem

Algorithmic calculations.

- Continued fractions and their use for simplifying fractions
- Converging fraction series for the golden section (see irrationals)
- Optional: the Euclidean algorithm for HCF and LCM with practical examples

Incommensurability in arithmetic and geometry.

- Extension of the number domain to include irrationals
- Extension of number series to include continued fraction approximations of roots
- Perhaps: root 1 to root 25 by continued fractions and their number series
- Squares and root 2
- Equilateral triangles and root 3
- Regular pentagon and root 5
- Derive the formula for pentagon sides and diagonals

GEOMETRY

- Revision of work with angle types
- Alternate segment theorem
- Using the triangle to revise congruence; similarity, proportional division using parallels; the four main centers of a triangle and the Euler line
- The forms of Pythagoras' Theorem, extension of the theorem
- Revision and deepening of area calculation (triangle, right angle, square, rhombus, parallelogram, trapezium, deltoid, area transformation)
- Investigation of the circle (circumference, area, π). Volumes of solids (cube, cuboid, prism, pyramid, cylinder, cone, sphere)
- Conics approached from locus as well as other curves (Cassini, Descartes). (Could also be done in Class 10.)
- A variety of plane solids or bodies usually through a main-lesson on descriptive geometry
- Diagonals
- Platonic and Archimedean solids
- Training in symmetries based on the simple Platonic solids cube, octahedron, tetrahedron, dodecahedron, icosahedron, and their duals
- Training in visualization of spaces by exercises in picturing to help drawing tasks, as well as separate description of spatial connections and construction methods
- Design of a simple alphabetic typeface
- Optional: The constant division of the golden section (application in architecture, nature and man) is also recommended for Class 10

Class 10

Points of view and general themes:

The pupils should be guided *"from knowledge to insight"* (R. Steiner). This means a completely new approach for the teaching method.

Trigonometry offers a wide field of activity for this. In the angle functions the pupils discover a completely new structure of relationships and also the practical use which can be got from this. The application of mathematical calculations should be experienced. This is possible through the parallel offered by physics (sine wave in statics, parabolas for projectiles), as well as the so-called 'surveying' project, which allows ideas to be shown in practice, where the pupils measure and draw a small ground area. Precision is learnt; the problems - not the teacher - correct the youngster. Similarly, in another main-lesson the pupils learn the special meaning of the normal technical projections. The various possibilities for creating pictures can be seen as the start for this. In continuation of perspective, spatial projection and some elements of projective geometry are worked through in drawing.

For arithmetic a final area is logarithms as the summit and final stage of this development.

Calculators are now used more frequently.

In algebra, quadratic equations are now dealt with, if not done already, and the various solution methods and formulae are developed.

Mathematics in Class 10 should have a strongly practical character. One can therefore take up the bookkeeping begun in Class 7, and move on to double-entry bookkeeping. *"The pupils become more aware of commercial processes and the way businesses develop, which means that they can no longer reflect on these things. They will notice the aspect of interdependence, and the implications of social responsibility become evident."* ¹²

Content suggestions:

As for Class 9, there is more material than can be dealt with here:

ALGEBRA

Quadratic equations

- The quadratic expansion
- Development and use of the general formula
- Development and proof of Vieta's root formula (principle points on a quadratic graph)

- Development of the general quadratic equation
- The significance of determinants
- Optional: linear and quadratic inequalities

Powers with whole and rational indices, logarithm

- Revision of methods with natural number indices Series with index of 2 or 3
- Extension of the index domain over rationals, integers and real numbers
- Development of logarithms and creation of a table base 2, 3, 10
- Calculation with logarithmic tables (briefly, for practice in use of tables)
- Commutativity, associativity, distributivity
- Rules of logarithms
- Solution of exponential equations Solution of logarithmic equations The logarithmic and exponential curves (first contact with functions)
- Optional: Logarithmic scales in science. Archimedean spirals, logarithmic spirals ' (morphological examples from nature. evolutes) The biography of Euler

Number Series (or possibly in Class 11)

- First contact with series, especially monotones. e.g. arithmetic, geometric, exponential. the Lukas and Fibonacci series
- Application to calculation of interest and various means

PLANE TRIGONOMETRY (application in the surveying project)

- Revision of similar triangles and proportional division by parallels and its extension to the image through concentric similar shapes
- Angle measuring systems: degrees, gradients, radians
- Sine, cosine, tangent (cotangent)
- Solution of basic exercises in a right triangle and extension to include plane figures and solids
Angle functions in circle of unity radius
- Use of angle functions in the general triangle by dividing into two right triangles
- Derivation of cosine rule (recognition of the special case of Pythagoras' Theorem)
- Use of these techniques for calculation in surveying
- Development of the sine rule
- Development of the trigonometrical area formula Graphical representation of angle functions

DESCRIPTIVE GEOMETRY

- Description of plane solids by various means Representation of solids
- Exercises in interpenetration of solids Curvilinear solids
- Shadow construction
- The boundary case of an infinite point in shadow construction
- Transformation of a dodecahedron into an icosahedron; stages of interpenetration
- Screws, snails, spirals
- Technical drawing: designs and details of pupils' own carpentry

GEOMETRICAL WORK ON CIRCLE AND LINE

- Proof of the construction of the golden section Construction of the golden section
- Perhaps: the golden section in the human being Dürer. Le Corbusier). See Class 9

ELEMENTS OF PROJECTIVE GEOMETRY

- The question of parallelism and the infinite point on a straight line.
- Optional: The circle as the curve of division; tackling problems of technology
- Periodic experience applied in mechanics in industry (e.g. linkages, drive machinery).
- Running a bank account book and diary Balance sheets and household budgeting Recording of a complete project
- Optional: Double-entry book-keeping project

Class 11

Points of view and general themes:

The areas of geometry and algebra which were dealt with separately until now are brought together in analytical geometry. It will become clear to the pupils how geometrical forms have their counterpart in equations and how new geometrical forms can be defined by equations. Straight lines are dealt with as the track of a movement and functions elaborated. Vectors will also be formally established following their introduction in physics in Class 10. How far vectors are taken depends on the teacher's assessment of the maturity of the class.

The laws of Euclidean geometry are raised to a new stage in projective geometry. Infinity should be grasped by working through the 'elements of infinity' (infinite point, line and plane). Through this the pupils acquire an extension of their thought space.

In a study of oscillation, trigonometry from Class 10 is brought into motion giving a basis for the wave theory background of radio communication (physics, Class 11).

With spherical trigonometry the student can experience an enhancement of plane trigonometry. Like analytical geometry, this gives a connection between arithmetic and geometry. As in many areas, Class 11 sees how earlier separate areas of study come together connections should be formed.

A new stage in thinking meets the pupils as they work with series towards the sum of an infinite series. In percentages, a new process is discovered as steps tend towards zero. Calculation of 'half lives' provides a link with atomic physics in Class 11, and this leads on to topical questions.

Again geometry should be explored in a separate main-lesson (timetable permitting) and the teacher can choose a selection of topics from the available range according to class ability.

Content suggestions:

SERIES AND SEQUENCES

- Introduction to finite and infinite arithmetic and geometric series and sequences (if not done in Class 10)
- Development of next term and sum of term, formula and applications in science and industry
- Perhaps: draw connections between part formula of higher order arithmetic series and binomial coefficients
- Achieve a clear limiting value and find out sum
- Graphic examples of geometric series
- Development of the compound interest formula as a special case of geometric series: application to a variety of problems from nature and business (half lives, growth patterns, etc.)
- Discovery of e

FUNCTIONS (OR IN CLASS 12)

- Leibniz's idea of functions
- Domain and range
- Graphs of functions
- First inverse functions

ALGEBRA AND ANALYTICAL GEOMETRY

- Historical context
- Introduction to Cartesian and polar co-ordinates and their connection
- Point, distance and line in the above systems
- Linear equations in their various forms and then plane graphs using co-ordinates
- Application of calculation to geometrical problem (intersection (if two lines, calculation of special points in a triangle)
- Introduction to vectors from physics I (if not done in Class 10)
- Discovery of the circle formula
- Investigation of positional relationship between circle and line (chord, tangent, external line; polar)
- Discovery of basis for the tangent equation and recognition of the significance of the determinant. By this means acquaintance with:
- Complex numbers as solutions to quadratic equations.
- Optional: Intersection angles of circle and line, separation into linear factors, discovery of formulae for ellipse, hyperbola, parabola, especially their tangents or asymptotes (can also be done in Class 12).

All solutions should be done both by calculation and graph!

OSCILLATIONS (mathematical basis for electricity main-lesson in Class 11)

- Revision of unity circle, gradients, radians
- Description of angle functions in Cartesian terms and also graphically
- Physical oscillations approached from mechanics point of view
- Perhaps: representation using polar co-ordinates, mathematical formulation of physical quantities (amplitude, frequency, wavelength, period, phase, frequency modulation, oscillation). Include the addition theorem, also algebraic formula of waves. Application: calculation and representation of 3-phase current.

Selected approach to projective geometry or spherical geometry depending the class's stage of maturity (possibly in Class 12)

PROJECTIVE GEOMETRY

- Infinite elements
- Concept of duality
- Desargues' Theorem
- Pascal's Theorem
- Basic harmonic figures, harmonic reflections

SPHERICAL GEOMETRY

- Pole and polar planes of the sphere, introduction to non-Euclidean geometry on a sphere
- The 'Parallel axiom'
- Graphical representation of great circles and small circles on a sphere Spherical bi-angles
- Calculation of the surface area and volume of a sphere
- Construction of a spherical triangle by means of 3 defined dimensions (congruence rules); angle sum in a spherical triangle
- Construction of tangents in the triangle points on a sphere

MATHEMATICAL GEOGRAPHY

- Calculation of great circle angle, great circle distance and course angle
- Development of the sine rule of spherical geometry
- Optional: the cosine rule of spherical geometry
- Geographically derived measurements (meter, nautical mile, knot)
- Polar triangle

MATHEMATICAL ASTRONOMY

- Graphical representation of horizontal system and stellar position in space
- The equatorial system (RA declination)
- Perhaps: construction of a nautical sextant the sun in the course of the year; calculation of time (local, time zone, sidereal; calculation of the calendar, the Platonic year, lunar and solar rhythms)
- Algebra of switches; the concept of 'or' and 'and' can be shown with switches in parallel and series
- Set theory
- Mathematical logic

Optional: Boolean algebra (introduction of information technology (can be done in other classes).

Class 12

Points of view and general themes:

This class should take an important further step from what was achieved in Class 11. Just as Class 11 moved in analytical geometry from visual into algebraic, the reverse is done in Class 12. In calculus the pupils should move from the purely numerical into an experience of differentiation and integration. Limits of series should be grasped as the representatives of an endless process. The pupils should understand this new dimension of mathematics by working with the concept of the 'difference quotient': the quotients of two difference series which both tend towards zero produces something completely new. This should not be merely applied but should be grasped thoroughly, and experienced. In calculus the equations should be made so translucent "that one gets a feeling for what is what in equations" ¹³. Only then should the sensually visible graphical form of the calculation be shown. Find the form from the equation, recognize the form of the equation. In this way one nurtures inner activity in the pupils and also an understanding for the functional

connections as well as the qualitative in mathematics, which is after all indispensable for a true understanding of current physics. In this connection it is also possible to show that functions of this type can be used decisively in applied physics; in optics, electricity, mechanics and motion. Quality and quantity part company.

While working through the basic rules of integrals, the pupils should recognize that there is also a mathematical method (differentiation), a polarity in higher mathematics which permits the mathematical grasp of the world to reach a new level.

Depending on what was done in Class 11, projective geometry can be built up from the central projection (perspective), or spherical geometry worked through. Projective geometry can be worked through in such a way that it opens a link to understanding applications in bird's eye perspective, and perspective drawing in the architecture main-lesson on a possible art trip. Spherical geometry can be approached from a more representational or analytical angle, and tend more towards astronomy or to the earth (also in the context of descriptive geometry).

A further possibility for a second mathematics main-lesson consists of bringing together the sciences of mathematics, botany, astronomy, embryology and geometry in a great picture. Attempting this would, however, depend very much on the maturity of the class.

Content suggestions:

INFINITESIMALS

- Revision of function concepts, building on the number continuum
- (independence of real numbers)
- A look at the turn of the eighteenth century with the historical development of infinitesimal calculations through Newton and Leibniz
- Extension of the connections between function and graph by means of elementary functions

CALCULUS

- Development of difference quotients Discovery of differential quotients
- Working through the rules of differentiation of polynomials, reciprocal functions, root functions and angle functions
- Derivatives as increasing functions and their expression in the concepts of speed and acceleration
- Product, quotient and chain rules
- Optional: derivatives as inverse functions
- The connection between the initial function and its derivative shown graphically
- Discussion of curves for polynomials of order 2, 3, 4, and reciprocal functions
- Application in various areas of work including technology
- The function of e , natural logarithms (possible extension of current knowledge of these)
- Definition of function terms from characteristics of their graphs
- Maxima and minima with exercises from industry, optics (Fermat's principle)

INTEGRALS

- Derivatives and the return to the original function as opposite mathematical processes
- Development of integrals for polynomials
- Integrals functions as a function of the upper limit of a particular integral
- The concept of original functions and 'indefinite integrals'

CORRECTIONS

After reading his article in *Waldorf Science Newsletter* #12, Günter Althage suggests the following changes might help the reader gain a clearer understanding of what he intended to say.

Page 8, line 2:

instead of "...by observing black and white objects" insert "by observing monochromatic objects"

Page 8, Fig. 1b

please eliminate fig. 1b this figure is not relevant here!

Page 9, line 13:

instead of "...yet he was incapable of penetrating this phenomena" insert "yet he didn't gain acceptance"

Page 10:

Starting line 5 the new text reads as follows:

He constructed the images of a rod lying horizontally in water (fig. 3) as well as one immersed vertically (fig. 4), both according to the law of elevation (see fig. 1a) and as well as according to the conventional construction (see fig. 2). The upper more distorted curve in fig. 3 and the curves 1, 2, 3, 4, 5 for the spectator positions A1 resp. A2, A3, A4, A5 in fig. 4 result from the conventional construction (fig. 2). The lower curve in fig. 3 and the vertical rod elevated in itself in fig. 4 result from the law of elevation (fig. 1a). After comparing the experience von Baravalle came to the conclusion that only construction of vertical elevation was capable of correctly reproducing the phenomena. Figures 3 and 4 are from Dr. Hermann von Baravalle "Erweiterung... der Perspektive, Krümmung des Sehraumes" [The widening of perspective, curvature of the visual space], Stuttgart. Fig. 3, Fig. 4

This publication contains references to further essays of E.A.K. Stockmeyer and H. von Baravalle on the subject of optical elevation. A brief description of elevation may also be found in "Physik als reine Phänomenologie", Bern 1951, Band III, page 130 ff. by von Baravalle (Fig. 6a and 6b).

Eliminate totally fig. 5a and 5b (they don't help) and replace them by the following sentence:

The figures 6a and 6b demonstrate the constructions of the apparent lifting (vertical elevation) of a rod lying horizontally in water and a rod immersed diagonally in water.

page 11, line 11

instead of "...to the construction (Fig. 7)" insert; to the construction (Fig. 2) Instead of "...drawings.

In Fig. 7 the elevated..." insert; "...drawings. In Fig. 7a the elevated..."

page 11, line 12: instead of

".....in the fig. 7b construction, " insert; " in the fig 7a construction,..."

page 11, line 13:

instead of "... given in Fig 7 is incomplete in itself" insert; "...given in Fig. 2 is incomplete in itself"

page 11, line 14:

instead of "by a fractured level plane" insert; "by a fracturing (??) level plane"

page 13, line 4/5:

instead of "... the pupil is oriented parallel " insert; "the pupil is orientated parallel instead of ..." elevation prevails (see Fig. 3 and 4) insert; "... elevation prevails (see Fig. 3, 4, 6a and 6b)

page 13, line 8:

instead of "... to the point of the eye (see Fig. 3 and 4, again) - .. " of the eye (see Fig. 3, 4, 6a and 6b, again)

page 17 the photo 1 on page 17

is reproduced upside down! It has to be rearranged the other way round and the following corrections in the text result from this:

page 17, line 4:

instead of "... is looking obliquely from below" insert; "is looking obliquely from above"

page 17, line 9

instead of "... one sees the mirror image of the submerged measuring rule reflected on the undersurface of the water. In addition the upper part of the rule is also seen refracted, through the upper surface of the water, as viewed from the side of the aquarium." insert "...one sees the mirror image of the measuring rule in the air reflected on the surface of the water. In addition the submerged part of the rule is also seen refracted through the upper surface of the water, is elevated very strongly, and furthermore it is viewed from the side of the aquarium.