PRINCIPAL COMPONENT VALUE AT RISK

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Let $\Pi(t)$ be the value at time t of some asset or portfolio of assets. The financial risk inherent to the portfolio is due to the future price fluctuations

$$\Delta\Pi(t, \delta t) = \Pi(t + \delta t) - \Pi(t) \quad (1)$$

These are of course unknown, and the best we can do in general is to modelize them using some probability distribution function (PDF). Traditionally in Finance, risk has been quantified by the variance or volatility of this PDF, but in recent years another measure of risk, the *Value at Risk* or VaR, has grown popular. Formally, it is defined as follows: given an $\alpha \in [0, 1]$, one defines the Value at Risk VaR_α by the (implicit) equation :

$$\text{Prob}\,(\Delta\Pi(t, \delta t) \leq -VaR_\alpha) = \alpha \quad (2)$$

(assuming (2) has a unique solution, which in practice usually is the case). The right probability to use here is the one conditional on information available at time t. VaR has been popularized by J. P. Morgan's RiskMetrics [7]. It's main appeal lies in it's transparent financial interpretation, which is that with probability $1 - \alpha$ one's financial loss at time $t + \delta t$ will be less than VaR_α.

Unlike variance, VaR does not treat losses and gains on on equal footing. On the other hand, for a normal distribution with mean 0, knowing VaR_α for any α amounts to knowing the variance. A similar remark applies to any one-parameter family of distributions whose members are obtained by scaling, like Student-t or symmetric Lévy-stable distributions, where the variance may be replaced by some other measure of the spread of the distribution (for example, in the case of Lévy-stable distributions, the tail parameter introduced in [2], or the expectation of a p-th power of the the relevant random variable for suitable $p < 2$).

VaR, as just defined, would appear to be a relatively uncontroversial concept. Some of the controversy surrounding it seems to be connected with the choice of PDF to model (1). A widely used model is the RiskMetrics model [7], a particular case of a GARCH(1, 1), in which the daily log-returns are normally distributed with time-dependend variance. At this Conference evidence has been presented on the inadequacies of such a model for financial asset returns, especially in connection with the occurrence of fat tails, a phenomenon which is of obvious importance for VaR_α-estimates for small α. However, there is also evidence that the normal model works reasonably well for VaR-estimates up to the (RiskMetrics') 95% confidence level : see [5], in particular pp 133 - 138, for a discussion of the available results up to 1997. The issue we want to address here is not that of fat tails, but rather that of non-linearity, and we will work in the traditional log-normal setting. Concerning the type of non-linearity we consider, we will work in the quadratic or Delta - Gamma approximation to the portfolio. We note here that, recently, Bouchaud and Potter [3] treated the similar problem for portfolio's for which there is a dominant risk-factors with non-Gaussian statistics.

A certain number of propositions for computing or approximating quadratic VaR in the normal or log-normal setting have been made in the litterature and are presently used : we mention the Delta-Gamma Normal approach, Wilson's Delta-Gamma Approach, the Higher Moment Delta-Gamma Approaches of Zangari (cf. [5]) and, more recently, the Fourier based method of Albanese and Seco [1]. Here we report on a new approximation, based on the asymptotics of Gaussian integrals over quadrics, which has the advantage of being completely explicit and easy to compute as well as being amenable to a precise error analysis.

Let Π be some investment portfolio depending on a vector of risk factors $(S_1(t), \cdots, S_n(t))$. The value of Π will be some function $\Pi(S_1, \cdots, S_n, t)$ of these riskfactors and of time (one may think of a portfolio of derivatives with the underlying as risk factors). We assume the S_j to have a multivariate log-normal distribution: $S_i(t+\delta t) = e^{r_i} S_i(t)$, with $r = (r_1, \cdots, r_n)$ jointly normally distributed with mean m and variance - covariance matrix $\mathbb{V}$. We fix t and assume that m and $\mathbb{V}$ are both proportional to δt. If we write the P& L-function $\Delta\Pi$ as a function of the return vector r : $\Delta\Pi = \pi(r_1, \cdots, r_n)$ (we will generally suppress the time-parameters t and δt from our notations) then the cumulative distribution function of $\Delta\Pi$ is given by

$$I(V) = \mathrm{Prob}\,(\Delta\Pi < -V) = \int_{\{\pi(r) < -V\}} e^{-(r-m, \mathbb{V}^{-1}(r-m))/2} \frac{dr}{(\det \ 2\pi\mathbb{V})^{1/2}} \tag{3}$$

and $VaR_\alpha = I^{-1}(\alpha)$. The problem of computing VaR thus amounts to the problem of efficiently computing and inverting the function $I(V)$.

In practice, numerical estimation of VaR is mostly done using Monte-Carlo methods. These are accurate, but computationally intensive, time-consuming and not very flexible. It is therefore of some interest to complement these by analytical methods which are able to provide some reliable " quick and dirty " estimates for the VaR. We start off with the Delta-Gamma approximation to our portfolio :

$$\pi(r) \simeq \Theta \cdot \delta t + (\Delta, r) + \frac{1}{2}(r, \Gamma r), \tag{4}$$

where

$$\Theta = \frac{\partial\Pi}{\partial t}, \quad \Delta_i = S_i \frac{\partial\Pi}{\partial S_i}$$
$$\Gamma_{ij} = S_i S_j \frac{\partial^2\Pi}{\partial S_i \partial S_j} + S_i^2 \frac{\partial\Pi}{\partial S_i} \delta_{ij},$$

everything evaluated at time t. We have collected in (4) all terms of order up to 1 in δt, since the expectation of r will be of the order of $\sqrt{\delta t}$ for small δt, given pour assumptions on m and $\mathbb{V}$. Note that if one uses percentage returns instead of log-returns, as is sometimes done in the litterature (for example [5]), the expression for Γ_{ij} changes. We now replace $\pi(r)$ in the integral (3) by it's quadratic approximation. This leads, after some easy transformations, to the following integral:

$$I(V) = \int_{Q(z-v) < -V-T} e^{-|z|^2/2} \frac{dz}{(2\pi)^{n/2}}, \tag{5}$$

where the quadratic form Q, the vector v and the constant T are obtained as follows: writing $\mathbb{V} = \mathbb{H}\mathbb{H}^t$ (if $\mathbb{H}$ is upper or lower diagonal this is known as a Cholesky decomposition),

$$Q(z) = \frac{1}{2}(z, \mathbb{H}^t \Gamma \mathbb{H} z)$$

and

$$v = \mathbb{H}^{-1}\Gamma^{-1}\Delta + \mathbb{H}^{-1} m, \quad T = \Theta - \frac{1}{2}(\Delta, \Gamma^{-1}\Delta)$$

The basic object here is Q, which can be interpreted as a sensitivity-adjusted covariance matrix. Note that $v = 0$ if both $m = 0$ (standard RiskMetrics' assumption) and $\Delta = 0$ (a Delta-hedged portfolio). In the paper [4] we derived the following asymptotic expansion:

Theorem 0.1. *Let*

$$J(R^2) = \int_{Q(x-v)<-R^2} e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$$

and suppose that Γ *is not positive definite (i.e. there is some Gamma risk present). Then*

$$J(R^2) \simeq e^{\gamma}\ e^{-R^2/2|a_{\min}|} \sum_{\nu\geq 0} C_\nu R^{-1-\nu} \tag{6}$$

as $R \to \infty$*, where* $a_{\min} < 0$ *is the smallest eigenvalue, or bottom of the spectrum, of* $\mathbb{Q} := \mathbb{H}^t \Gamma \mathbb{H}$ *:*

$$a_{\min} = \inf_{|x|=1} (\mathbb{Q}x, x),$$

which we assume to be simple. The constants $\gamma = \gamma(v)$ *and* C_0 *can be computed as follows: if* a *is any eigenvalue of* $\mathbb{Q}$ *(repeated according to it's multiplicity), let* v_a *denote an associated normalized eigenvector (to be chosen mutually orthogonal if* a *has multiplicity greater than* 1*). Then*

$$\gamma = -\frac{1}{2}\left(\sum_a{}' \frac{a}{a - a_{\min}} (v, v_a)^2 + (v, v_{a_{\min}})^2\right) \tag{7}$$

and

$$C_0 = 2(2\pi)^{n/2-1}\ \frac{|a_{\min}|^{n/2}}{\Pi' |a - a_{\min}|^{1/2}} \tag{8}$$

Here $\sum'$ *and* Π' *mean the sum and product, respectively, over all eigenvalues different from* a_{min}

This expansion is relevant for VaR_α-estimations for small α. The assumption that $a_{\min}$ is simple is not essential: one can even give an asymptotic expansion which is uniform in the first few eigenvalue differences: cf. [4]. The coefficients C_ν can in principle all be computed, but in practice one would rarely go beyond the main term. It is also possible to give precise error-estimates. There is an interesting difference between the cases $v \neq 0$ and $v = 0$: in the latter case all coefficients C_ν with odd index vanish, and the asymptotic expansion becomes one in negative powers of R^2. Finally we note that the methods we used to prove the theorem extend to non-quadratic portfolio's.

The coefficient γ can also be written as follows : if $P_{\min}$ denotes the orthogonal projection onto the minimal eigenvector $v_{\min}$, and $P' = Id - P_{\min}$, then (7) is the same as

$$2\gamma = -\left(v, P'(\mathbb{Q} - a_{\min})^{-1}\mathbb{Q}P'v\right) - |P_{\min}(v)|^2$$

This shows that we need not diagonalize the whole matrix $\mathbb{Q}$, just find $a_{\min}$, $v_{\min}$ and invert the matrix $(\mathbb{Q} - a_{\min})$ on the orthogonal complement of $v_{\min}$: also note that the denominator in the formula (8) for C_0 is simply the square root of the determinant of $P'(\mathbb{Q} - a_{\min})P'$). These remarks simplify the numerics.

Armed with this this expansion, we return to VaR. If $V + T > 0$, then, by (5) and (6), limiting ourselves to the main term of the expansion, quadratic VaR_α will be approximated by the (unique) solution of the equation

$$(V+T) + |a_{\min}|\log(V+T) = 2|a_{\min}|\left(\log(\alpha^{-1}) + \gamma + \log C_0\right)$$

which can easily be found numerically.

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