

Margin calls, fire sales, and information constrained optimality

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Abstract:

Protection buyers share risk with protection sellers, whose assets are only imperfectly pledgeable because of moral hazard. To mitigate moral hazard, privately optimal contracts involve variation margins. When margins are called, protection sellers must sell some of their assets to other investors. We analyse, in a general equilibrium framework, whether this leads to inefficient fire sales. If markets are complete, so that investors buying in a fire sale interim can also trade ex ante with protection buyers, equilibrium is information-constrained efficient. Otherwise, privately optimal margin calls and asset sales are inefficiently high. To avoid inefficient firesales, public policy should facilitate ex-ante contracting among all relevant counterparties.

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1 Introduction

One of the major functions of financial markets is to enable participants to share risk. For example, banks desiring to hedge their holdings of mortgages, loans or bonds can trade in derivative markets, such as, e.g., the CDS market. The risk-sharing effectiveness of such trades, however, can be significantly reduced by counterparty risk. For example, when Lehman Brothers filed for bankruptcy in September 2008, it froze the positions of more than 900,000 derivative contracts (Fleming and Sarkar, 2014). Margins can help mitigate this risk. Initial margins, i.e., collateral that must be deposited at the inception of a risky position, have been studied by, e.g., Gromb and Vayanos (2002) or Kuong (2014). The present paper focuses on *variation* margins, which are also frequently used, as described by McDonald and Paulson (2015, p 92):

“As time passes and prices move ... fair value will [become] positive for one counterparty and negative ... for the other; in such cases it is common for the negative value party to make a compensating payment to the positive value counterparty. Such a payment is referred to as *margin* or *collateral*... payments due to market value changes are variation margins... this transfer of funds based on a market value change is classified as a change in collateral and not as a payment. The reason is that the contract is still active so collateral is held by one party against the prospect of a loss at the future date when the contract matures or makes payment on a loss. If the contract ultimately does not generate the loss implied by the market value change, the collateral is returned.”

As explained by McDonald and Paulson (2015), variation margin requirements are negotiated *ex ante* between protection buyers and protection sellers, and written down in the contract between the two counterparties. For example, for the CDS written by AIG, McDonald and Paulson (2015, p 93) report that

“AIG would make collateral payments only if the decline in the value of

the insured assets exceeded some predefined threshold. These thresholds often depended on AIG’s credit ratings.”

Following the crisis, regulators and law-makers promoted a significant expansion of the use of margins in derivative activity (Dodd-Frank Act in the US, EMIR in the EU). Concerns were expressed, however, that variation margins could be procyclical and trigger fire sales (e.g., Bank for International Settlements, 2010).¹ To shed some light on these issues, the present paper analyses the welfare consequences of variation margins in general equilibrium, with micro-founded constraints. Taking a general equilibrium approach, with several markets and several types of agents, enables one to analyse interactions, in particular pecuniary externalities, across markets and across agents. Considering an environment in which all agents optimise, in particular when designing contracts, enables one to analyse welfare.

Model: Our model features three types of agents:

- First, there is a mass-one continuum of risk-averse agents endowed with a risky asset. These risk-averse agents seek to hedge the risk of their asset, and we refer to them as “protection buyers.” Protection buyers can for example be commercial banks, and their assets can be mortgage loans or bonds.
- Second, there is a mass-one continuum of risk-neutral agents. Since these risk-neutral agents are the natural providers of insurance for the protection buyers, we refer to them as “protection sellers.” Protection sellers have limited liability but hold assets whose output can be used to make insurance payments. Protection sellers can, e.g., be broker dealers or specialised firms such as AIG.² Their assets can be portfolios of loans or trading strategies. Protection sellers must exert effort to ensure their assets

¹Bignon and Vuillemeys (2018) offer an interesting historical example of this type of concern.

²Duffie et al (2015) empirically analyse the risk exposure of broker-dealers and their customers in the CDS market. These broker dealer (which are often subsidiaries of large investment banks) are a good example of the protection sellers in our models, while their customers are a good example of the protection buyers in our model.

are profitable. In the case of loans this involves loans monitoring and servicing. In the case of trading strategies, this involves transactions costs minimisation and appropriate management of collateral and custody. Moreover, in both cases, effort is also needed to prevent operational risk.

- Third, there is a mass-one continuum of risk-averse agents, each endowed with one unit of safe asset. Investors can also manage the assets held by the protection sellers, but they are less efficient at that task than the protection sellers. Investors can be thought of as hedge funds, investment funds or sovereign wealth funds. It is natural to assume that they are less efficient at monitoring loans or managing trading strategies than the originators of those loans and strategies.

As in Biais, Heider and Hoerova (2017), protection sellers' effort is unobservable, so there is a moral-hazard problem.³ After initial contracting, but before effort, a signal on the value of the assets of the protection buyers is publicly observed. When this signal reveals bad news about the protection buyers' assets, this increases the expected liabilities of protection sellers who sold insurance to protection buyers. The corresponding debt overhang reduces the protection sellers' incentives to exert effort.⁴ To maintain the protection sellers' incentives, insurance after bad news must be limited. The larger the amount of assets left under the management of protection sellers, the larger their cost of effort, the more severe the moral hazard problem. It is optimal, therefore, to reduce the amount of assets left under the management of protection sellers after bad news, in order to relax incentive constraints. To achieve this, the assets are transferred from the hands of the protection sellers to those of the investors.

³Biais, Heider and Hoerova (2017) offer a partial equilibrium analysis, with one protection buyer and one protection seller, and with an exogenous liquidation value for the protection seller's asset. In contrast, the present paper offers a general equilibrium analysis, in which the price of the protection sellers' assets is endogenous, and there are interactions between three different types of agents and markets.

⁴Bolton and Oehmke (2015) use a similar moral hazard framework to analyse whether derivative contracts should be privileged in bankruptcy. The seminal contribution on debt overhang is Myers (1977).

Contribution: The first contribution of the present paper is to analyse the incentive constrained Pareto set, i.e., the second best. It is the set of consumptions for the three types of agents, and asset transfers from protection sellers to investors, contingent on all publicly observable information, subject to incentive, participation and resource constraints. In the second best, the marginal rate of substitution between consumption after good news and consumption after bad news is equalised across protection buyers and investors, reflecting that they share risk optimally. In contrast, this marginal rate of substitution is different for protection sellers, reflecting that moral hazard prevents first-best efficient risk-sharing between protection sellers and protection buyers. Increasing the transfers of assets after bad news, from protection sellers to investors, improves risk sharing by relaxing incentive constraints, but generates productive inefficiencies. In the second best, asset transfers are such that the marginal benefit of improved risk-sharing is equal to the marginal cost of productive inefficiencies.

The second contribution of the present paper is to analyse competitive equilibrium in this environment. Our main focus is on complete markets, in which securities or contracts contingent on all observable variables can be traded, subject to incentive and participation constraints. Optimal contracts between protection buyers and protection sellers involve variation margins: After bad news, protection sellers must liquidate a fraction of their assets and deposit the proceeds on their margin accounts at the CCP. When such liquidations occur, the protection sellers' assets are purchased by the investors. Since the latter are less efficient than the protection sellers at managing their assets margin calls trigger price drops, which can be interpreted in terms of firesales (see Shleifer and Vishny, 2011).⁵ This generates pecuniary externalities: When one protection seller liquidates some of his assets to respond to margin calls, he contributes to depressing the price, which generates a negative externality on the other protection sellers, also selling at that price.

⁵As written in Shleifer and Vishny, 2011: “a fire sale is essentially a forced sale of an asset at a dislocated price... The price is dislocated because the highest potential bidders are ... selling similar assets themselves. Assets are then bought by nonspecialists who, knowing that they have less expertise with the assets in question, are only willing to buy at valuations that are much lower.”

The third contribution of the present paper is to analyse whether these pecuniary externalities imply that competitive equilibrium is constrained inefficient. We show that, with complete markets, equilibrium is incentive-constrained Pareto optimal. The intuition is the following. The decline in price triggered by variation margin calls generates profit opportunities for investors. Thus, while a negative signal is a negative shock for protection buyers, it is a positive shock for investors. This opens the scope for risk-sharing between investors and protection buyers. When the market is complete, investors and protection buyers fully exploit this risk-sharing opportunity, until their marginal rates of substitution are equalised, as in the second best. In contrast, when the market is incomplete, because investors cannot contract with protection buyers, equilibrium is inefficient, margin calls are excessively high, and firesales are inefficiently large.

Literature: Stiglitz (1982), Greenwald and Stiglitz (1986) and Geanakoplos and Polemarchakis (1986) show that, when markets are incomplete, equilibrium is constrained inefficient, while Prescott and Townsend (1984) analyse economies in which, in spite of asymmetric information or moral hazard, equilibrium is constrained efficient. Our paper talks to the two sides of the debate.

Recent contributions in which equilibrium is constrained inefficient consider risk-sharing (Gromb and Vayanos, 2002) or borrowing and investment (Lorenzoni, 2008):⁶ The seminal analysis of Gromb and Vayanos (2002) analyses how financially constrained arbitrageurs supply insurance to hedgers, who cannot directly trade with one another, so that markets are incomplete. Combined with financial constraints, market incompleteness generates pecuniary externalities implying that equilibrium is information-constrained inefficient. Lorenzoni (2008) considers entrepreneurs borrowing to fund investment projects. Because entrepreneurs are financially constrained they must sell assets after negative shocks. Moreover, markets are incomplete so that entrepreneurs cannot insure against negative shocks. Again,

⁶In his analysis of borrowing and investment, Kuong (2014) shows there exist multiple, Pareto ranked, equilibria, but does not analyse if (the best) equilibrium is information constrained efficient.

the combination of these two imperfections imply that information is constrained inefficient.⁷

While our result that, when markets are incomplete, equilibrium is constrained inefficient is in line with Stiglitz (1982), Greenwald Stiglitz (1986), Geanakoplos and Polemarchakis (1986), Gromb and Vayanos (2002) and Lorenzoni (2008), our result that equilibrium is constrained efficient when markets are complete is more in line with Prescott and Townsend (1984). (See also Kilenthong and Townsend (2014) and Kocherlakota (1998).) Prescott and Townsend (1984)'s main focus is on adverse selection, which is beyond the scope of our paper, but they also consider moral hazard. In their analysis of moral hazard, Prescott and Townsend (1984) consider a mass-one continuum of identical agents, each exposed to the risk of a loss, who can exert unobservable effort to reduce the probability of loss. Individual losses are i.i.d., and there is no macro-shock. In that context competitive equilibrium under moral hazard is information constrained. A distinguishing feature of our model, setting it apart from Prescott and Townsend (1984) and Kocherlakota (1998), is that a public signal, observed interim, can tighten incentive constraints, trigger variation margin calls and induce pecuniary externalities. The welfare analysis of this chain of events is a key contribution of our paper.

This one type of agent, one-market economy without macro-risk differs from the economy we consider, in which there are interactions across markets and agents' types and in which there is macro risk. Our setting enables us to identify the pecuniary externality and the risk-sharing and productive inefficiencies generated by moral hazard, which are not analysed by Prescott and Townsend (1984).

Davila and Korinek (2017) also offer an interesting analysis of the efficiency properties of competitive equilibrium when some agents are constrained. Our model and theirs are quite different, however. In Davila and Korinek (2017), when there is a complete set of Arrow securities, marginal rates of substitutions are equalised across agents. In contrast,

⁷Acharya and Viswanathan (2011) also conduct an equilibrium analysis of the price at which borrowers resell their assets in fire-sales. The incentive compatibility condition in our analysis is similar to theirs. Major differences between our paper and theirs are that i) we focus on risk-sharing while they focus on investment and borrowing, and ii) we conduct a normative analysis, studying whether equilibrium is constrained efficient.

in our model, even when there is a complete set of Arrow securities, moral hazard prevents equalisation of marginal rates of substitutions. Moreover, their analysis, in Application 1 of their paper, of information-constrained firesales differs from ours, in particular because in their analysis the financial constraint does not depend on prices, while it endogenously does in our analysis.

Outline: In section 2 we present the model. In Section 3 we analyse the first best, which can be attained when effort is observable. In Section 4 we analyse the second best, which can be attained when effort is unobservable, i.e., when there is moral hazard. In Section 5 we analyse equilibrium under moral hazard and show that it is information-constrained efficient when markets are complete, and inefficient otherwise. In Section 6 we discuss the implementation of equilibrium with real world financial instruments and offer empirical and policy implications.

2 Model

We consider an economy with three dates: time 0, time 1 and time 2; three types of agents: protection buyers, protection sellers and investors; one consumption good, consumed at time 2; and three types of asset generating output in consumption good at time 2.

Agents and endowments: There is a unit mass continuum of protection buyers, each with utility u , increasing and concave: $u' > 0$, $u'' < 0$, and endowed at time 0 with one unit of risky asset, paying $\tilde{\theta}$ units of consumption good at time 2. There is also a unit mass continuum of investors each with utility v , also increasing and concave, $v' > 0$, $v'' < 0$, and endowed at time 0 with one unit of safe asset paying 1 unit of consumption good at time 2. Finally, there is a unit mass continuum of risk-neutral protection sellers, each endowed with one unit of a productive asset, with potential payoff R at time 2.

Assets payoffs: The exogenous realisation of the protection buyers's asset at time 2 is denoted by $\tilde{\theta}$. $\tilde{\theta}$ can be take two values: $\bar{\theta}$ with probability π , or $\underline{\theta}$ with probability $1 - \pi$.

Each unit of the protection sellers' asset yields R at time 2 for sure if its holder exerts effort, at cost ψ per unit, at time 1. When consuming c_S units of the consumption good and exerting effort over y units of asset, a protection seller obtains utility $c_B - y\psi$. If the protection seller does not exert effort, his asset generates R with probability μ and 0 with probability $1 - \mu$. We assume $R - \psi > \mu R$, so that protection seller's effort is efficient.

Counterparty risk: That the risk averse protection buyers own a risky asset creates the scope for risk-sharing with the risk-neutral protection sellers. Protection sellers, however, have limited liability. To make insurance payments to the protection buyers, they must use the payoff from the assets they hold. If they don't exert effort, their asset's payoff can be zero. In that case, they can't pay the insurance promised to the protection buyer, which exposes protection buyers to counterparty risk. To ensure protection buyers are not exposed to counterparty risk, protection sellers must exert effort.

Signals: While output and consumption occur at time 2, at time 1 an advanced signal \tilde{s} on $\tilde{\theta}$ is publicly observed, before effort is exerted. When the final realisation of $\tilde{\theta}$ is $\bar{\theta}$, the signal is \bar{s} with probability λ and \underline{s} with probability $1 - \lambda$. When the final realisation of $\tilde{\theta}$ is $\underline{\theta}$, the signal is \bar{s} with probability $1 - \lambda$ and \underline{s} with probability λ . $\lambda > \frac{1}{2}$, so that the signal is informative.

Asset transfers: Effort takes place at time 1, after the signal is publicly observed. Before effort is exerted (but also after observing signal s), a fraction α of the productive asset can be transferred from the protection sellers to the investors. This is costly, however, because investors are less efficient than protection sellers at managing assets: Whatever α , the investors' per-unit cost of handling the asset is larger than that of the protection sellers': $\psi_I(\alpha) > \psi, \forall \alpha$. When consuming c_I units of the consumption good an exerting effort over α

units of asset, an investor obtains utility $v(c_I - \alpha\psi_I(\alpha))$. We assume the investor's per-unit cost of handling the asset is non-decreasing, $\psi'_I \geq 0$, and convex, $\psi''_I \geq 0$. Thus, investors' marginal cost, $\psi_I(\alpha) + \alpha\psi'_I$, is increasing. Yet, we assume it is efficient that investors exert effort when holding one unit asset: $R - \psi_I(1) \geq \mu R$. We also maintain the following assumption:

$$\psi_I(1) + \psi'_I(1) > \frac{\psi}{1 - \mu} > \psi_I(0). \quad (1)$$

As will be seen below, the right-inequality in (1) will allow for asset transfers, by making those transfers not too inefficient when α is close to 0. At the same time, the left-inequality in (1) will preclude full transfer of assets ($\alpha = 1$) because that would be too inefficient.⁸

Sequence of events: Summarising, the sequence of events is the following:

- At time 0, agents receive their endowments.
- At time 1, first the signal s is observed, then a fraction $\alpha(s)$ of the productive asset can be transferred from the protection sellers to the investors, and third agents decide whether to exert effort or not.
- At time 3, the output of the assets held by the protection buyers, the investors and the protection sellers is realised and publicly observed, and consumption takes place.

For given effort decisions, $\tilde{\theta}$ and \tilde{s} are independent of the output of the productive asset. When effort is exerted this trivially holds, since the output of the productive asset is a constant. When effort is not exerted, the output can be R or 0 , but this random variable is independent of $\tilde{\theta}$ and \tilde{s} .

⁸In general, assets could also be transferred to protection buyers. For simplicity we assume this is not possible as protection buyers don't have the technology to manage those assets.

3 First best

For simplicity we assume that the social planner places no weight on the risk neutral protection sellers, i.e., $\omega_S = 0$. Correspondingly, when analysing market equilibrium, we will assume zero bargaining power for the protection sellers. This simplifying assumption can be thought of as a normalisation and does not qualitatively affect our results.⁹ The Pareto weights on the protection buyer and the investors are denoted by ω_B and ω_I , respectively.

In the first best effort is observable and, since it is efficient, it is always requested by the planner and implemented by the agents. Hence the protection sellers' assets always yield R . The state variables on which decisions and consumptions are contingent are the publicly observable realisations of the protection buyer's asset (θ) and the signal (s). The social planner chooses the consumptions of protection buyers ($c_B(\theta, s)$), protection sellers ($c_S(\theta, s)$) and investors ($c_I(\theta, s)$), as well as how much asset to transfer from protection sellers to the investors ($\alpha(s)$), to maximise the weighted average expected utility

$$\omega_B E[u(c_B(\theta, s))] + \omega_I E[v(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))], \quad (2)$$

subject to the participation constraint of protection buyers

$$E[u(c_B(\theta, s))] \geq E[u(\theta)], \quad (3)$$

the participation constraint of investors

$$E[v(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))] \geq v(1), \quad (4)$$

⁹Unlike in the first best, when effort is unobservable, protection sellers are agents, while protection buyers are principals. Our assumption that protection buyers have all the bargaining power is in line with the principal agent literature in which the principal makes a take it or leave it offer to the agent.

the participation constraint of protection sellers

$$E[c_S(\theta, s) - (1 - \alpha(s))\psi] \geq R - \psi, \quad (5)$$

the budget constraint

$$c_B(\theta, s) + c_I(\theta, s) + c_S(\theta, s) \leq \theta + 1 + R, \quad \forall(\theta, s), \quad (6)$$

and the constraint that $\alpha(s)$ must be between 0 and 1. The participation constraints reflect the autarky payoff of protection buyers ($E[u(\theta)]$), investors ($v(1)$) and protection sellers ($R - \psi$), respectively. Solving this maximisation problem, we obtain our first proposition.

Proposition 1 *In the first best, there is no transfer of the productive asset, i.e., $\alpha(s) = 0, \forall s$, protection buyers and investors receive constant consumption, $c_B(\theta, s) = c_B, c_I(\theta, s) = c_I$. Their total consumption is*

$$c_B + c_I = E[\theta] + 1. \quad (7)$$

while the protection sellers' consumption is

$$c_S(\theta, s) = \theta - E[\theta] + R, \forall(\theta, s). \quad (8)$$

The efficiency achieved in the first best concerns both production and risk-sharing. Regarding production: The productive asset is entirely held by its most efficient holders, the protection sellers, i.e., $\alpha(s) = 0$. Regarding risk-sharing: All risk is borne by the risk neutral protection sellers, so that the risk averse agents (protection buyers and investors) bear no risk. As stated in (7), their, constant, total consumption is equal to the expectation of the output of the assets they are initially endowed with.

If there is a future market on $\tilde{\theta}$, the competitive equilibrium in which the future price is $E[\theta]$ implements the point on the Pareto frontier such that $c_B = E[\theta]$ and $c_I = 1$. In

that case, investors don't participate in the market and obtain their autarky utility, while protection sellers fully insure protection buyers at an actuarially fair price.

4 Second best

4.1 Moral hazard and pledgeable income

In the second best, protection sellers' effort is unobservable. Since the risk-neutral protection sellers have limited liability, there is a moral hazard problem. In that context, the social planner still chooses consumptions and asset transfers to maximise weighted average expected utility (2) under participation and budget constraints (3), (4), (5) and (6), but he must also take into account the incentive compatibility constraint that protection sellers must prefer effort than shirking (for simplicity we assume away incentive problems for investors).

Zero output for the productive asset is not possible under effort and thus reveals shirking. Hence, it is optimal to give zero consumption to the limited-liability protection sellers when output is 0. Thus, the incentive compatibility condition in state s is:

$$E[c_S(\theta, s) - (1 - \alpha(s))\psi | s] \geq \mu E[c_S(\theta, s) | s].$$

The left-hand side is the equilibrium-path expected consumption net of effort cost for the protection seller. The right-hand side is the off-equilibrium-path expected consumption of the protection seller when shirking, which is equal to the probability μ that the asset yields R in spite of shirking, multiplied by the expected consumption of the protection seller conditional on output R . The incentive compatibility condition rewrites as

$$E[c_S(\theta, s) | s] \geq \frac{1 - \alpha(s)}{1 - \mu} \psi, \tag{9}$$

The left-hand side of (9) is the equilibrium-path expected consumption of the protection seller conditional on s . (9) states that it must be large enough to make effort attractive.

Intuitively, this reduces the scope for risk sharing: To provide insurance against the risk associated with $\tilde{\theta}$, the protection seller must pay the protection buyer in state $\underline{\theta}$. This insurance payment, however, reduces the consumption of the protection seller in state $\underline{\theta}$. The right-hand-side of (9) is decreasing in $\alpha(s)$. The greater the fraction of the asset that is transferred from the protection seller to the investor, the lower the effort cost for the protection seller, the easier it is to satisfy the incentive compatibility constraint.

As in Holmstrom and Tirole (1997), for each unit of productive asset held by the protection seller, instead of the total return R , only the pledgeable income

$$\mathcal{P} \equiv R - \frac{\psi}{1 - \mu}, \quad (10)$$

can be promised without jeopardising incentives. We hereafter maintain the assumption that pledgeable income is positive, i.e., $\mathcal{P} \geq 0$. If \mathcal{P} was very large, then the first best could be achieved in spite of moral hazard. More precisely, it can be shown that if $\mathcal{P} \geq E[\theta] - E[\theta|\underline{s}]$ then the first best allocation is incentive compatible. To focus on the interesting case, we hereafter assume the opposite, i.e.,

$$\mathcal{P} < E[\theta] - E[\theta|\underline{s}]. \quad (11)$$

4.2 Risk sharing in the second best

The first order conditions with respect to the consumptions of protection buyers, protection sellers and investors yield the next lemma.

Lemma 1 *The consumption of the protection buyers and the investors only depends on the realisation of the signal, and not on that of θ .*

Lemma 1 means that protection buyers and investors are only exposed to the risk associated with s , which we hereafter refer to as signal risk. Correspondingly, we can omit θ in the consumptions of the protection buyers and investors, which we hereafter denote by

$(c_B(\bar{s}), c_B(\underline{s}))$ and $(c_I(\bar{s}), c_I(\underline{s}))$, respectively. The economic mechanism underlying Lemma 1 can be intuitively described as follows. As can be seen from (9), only the expected consumption of the protection seller conditional on the signal matters for incentives. For a given $E[c_S(\theta, s)|s]$, the split between $c_S(\bar{\theta}, s)$ and $c_S(\underline{\theta}, s)$ does not affect the incentive constraint nor the participation constraint of the protection seller. Hence it is optimal to design $c_S(\bar{\theta}, s)$ and $c_S(\underline{\theta}, s)$ to equalise the protection buyer's marginal utility in states $(\bar{\theta}, s)$ and $(\underline{\theta}, s)$. Similarly, it is optimal to equalise the investor's marginal utility in these two states.

Manipulating the first order conditions and using assumption (11), we obtain the next lemma, which is helpful for the characterisation of the second best:

Lemma 2 *i) In the second best, the resource constraint binds ($\lambda(\theta, s) > 0$) as well as the participation constraint of the protection seller ($\lambda_S > 0$). ii) One, and only, of the two incentive compatibility conditions (after \bar{s} and after \underline{s}) binds.*

In Lemma 2, i) simply reflects that no gains are left unexploited, while ii) reflects that if both incentive compatibility conditions were slack the first best would obtain, which is ruled out by assumption (11). Building on the above results, we can now state the following important characteristics on the second best outcome:

Lemma 3 *After a good signal, the incentive compatibility condition of the protection seller is slack and there is no asset transfer, i.e., $\alpha(\bar{s}) = 0$. After a bad signal, the incentive compatibility condition of the protection seller binds. Moreover, the consumption of protection buyers is larger after a good signal than after a bad signal, i.e., $c_B(\bar{s}) > c_B(\underline{s})$.*

When a bad signal \underline{s} is observed, this raises the probability that state $\underline{\theta}$ will occur, so that protection sellers are more likely to make insurance payments to protection buyers. This reduces the expected consumption of protection sellers, which, by (9), tightens the incentive compatibility condition. In contrast, after good news (\bar{s}), the expected consumption of protection sellers is large, so that the incentive compatibility condition is slack. Asset

transfers after bad news, $\alpha(\underline{s})$, relax incentive problems by lowering the right-hand-side of (9). In spite of asset transfers, however, the first best cannot be achieved. Hence, protection buyers keep some exposure to signal risk, so that $c_B(\bar{s}) > c_B(\underline{s})$.

To complete the analysis of risk sharing in the second best for given $\alpha(\underline{s})$, we now characterise the consumption of the protection sellers and protection buyers after good and bad signals. To do so, we proceed in three steps:

- First we observe that full risk sharing conditional on the signal (Lemma 1) and binding resource constraints (Lemma 2) imply $c_S(\theta, s) = \theta + 1 + R - (c_B(s) + c_I(s))$. Taking expectations conditional on each of the two possible realisations of the signal yields:

$$E[c_S(\theta, s)|\underline{s}] = E[\theta|\underline{s}] + 1 + R - (c_B(\underline{s}) + c_I(\underline{s})), \quad (12)$$

and

$$E[c_S(\theta, s)|\bar{s}] = E[\theta|\bar{s}] + 1 + R - (c_B(\bar{s}) + c_I(\bar{s})). \quad (13)$$

Using (12) and (13), the binding participation constraint of the protection sellers, and their binding incentive constraints after bad news, we obtain the total consumption of the protection buyers and investors after each of the two possible realisations of the signal.

- Second, we note that Lemma 1, and the first order conditions with respect to the consumption of protection buyers $c_B(\theta, s)$ and that of investors $c_I(\theta, s)$ pin down the split of consumption between them.
- Third, we remark that protection buyers and investors are only exposed to signal risk and are able to share risk optimally, because there is no information asymmetry or incentive problem among them. Correspondingly, the first order conditions with respect to $c_B(\theta, s)$ and $c_I(\theta, s)$ imply that the marginal rates of substitution of the protection buyers and investors between the two possible values of the signal are equalised.

Combining these results, the next proposition characterises consumptions of the protection buyers and investors for a given level of asset transfers $\alpha(\underline{s})$.

Proposition 2 *After a bad signal the total consumption of the protection buyers and the investors is*

$$c_B(\underline{s}) + c_I(\underline{s}) = 1 + E[\theta|\underline{s}] + \alpha(\underline{s})R + (1 - \alpha(\underline{s}))\mathcal{P}, \quad (14)$$

while after a good signal it is

$$c_B(\bar{s}) + c_I(\bar{s}) = 1 + E[\theta|\bar{s}] - \frac{Pr[\underline{s}]}{Pr[\bar{s}]}[\alpha(\underline{s})(R - \psi) + (1 - \alpha(\underline{s}))\mathcal{P}]. \quad (15)$$

Signal risk is perfectly shared between protection buyers and investors

$$\frac{v'(c_I(\underline{s}) - \alpha(\underline{s})\psi_I(\underline{s}))}{v'(c_I(\bar{s}) - \alpha(\bar{s})\psi_I(\bar{s}))} = \frac{u'(c_B(\underline{s}))}{u'(c_B(\bar{s}))}. \quad (16)$$

Finally, after each signal realisation, the split of consumption between protection sellers and investors reflects their Pareto weights:

$$\frac{u'(c_B(s))}{v'(c_I(s) - \alpha(s)\psi_I(\alpha(s)))} = \frac{\omega_I + \lambda_I}{\omega_B + \lambda_B}. \quad (17)$$

As long as the participation constraints of the protection buyers and investors don't bind (so that $\lambda_I = \lambda_B = 0$), the optimal allocation (for a given level of asset transfers, $\alpha(\underline{s})$) is fully characterised by four variables: $c_B(\underline{s})$, $c_B(\bar{s})$, $c_I(\underline{s})$, and $c_I(\bar{s})$. These four variables are pinned down by the four equations in Proposition 2. To interpret the total consumption of protection buyers and investors in the second best, it is useful to compare (14) and (15), their counterparts in the first best:

$$\{c_B(\underline{s}) + c_I(\underline{s}) = 1 + E[\theta], \quad c_B(\bar{s}) + c_I(\bar{s}) = 1 + E[\theta]\}, \quad (18)$$

their counterparts under moral hazard and contracting at time 1:

$$\{c_B(\underline{s}) + c_I(\underline{s}) = 1 + E[\theta|\underline{s}], c_B(\bar{s}) + c_I(\bar{s}) = 1 + E[\theta|\bar{s}]\}, \quad (19)$$

and their counterparts under moral hazard but no asset transfers

$$\left\{ c_B(\underline{s}) + c_I(\underline{s}) = 1 + E[\theta|\underline{s}] + \mathcal{P}, c_B(\bar{s}) + c_I(\bar{s}) = 1 + E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]} \mathcal{P} \right\}. \quad (20)$$

Comparing (18) to (19), one can see that in the former the same unconditional expectation appears after both signals, reflecting full insurance, while in the latter different conditional expectations appear, with $E[\theta|\underline{s}] < E[\theta|\bar{s}]$, implying lower total consumption for investors and protection buyers after bad news, i.e, signal risk. Comparing (19) to (20), one can see that signal risk is mitigated by pledgeable income. Relative to (19), after bad news, the total consumption of protection buyers and investors in (20) is increased by \mathcal{P} , and, to ensure that protection sellers break even, after good news their total consumption is decreased by $\frac{\Pr[\underline{s}]}{\Pr[\bar{s}]} \mathcal{P}$. Note that, under assumption (11), relying on pledgeable income is not sufficient to restore the first best, since (11) implies that

$$1 + E[\theta|\underline{s}] + \mathcal{P} < 1 + E[\theta].$$

When asset transfers are not used, protection sellers can only pledge to pay \mathcal{P} after bad news. Asking them to make larger insurance payments would jeopardise their incentives to exert effort. Relying on asset transfers can enable to provide more insurance, however. Comparing (20) to (14), one can see that, with asset transfers, an additional payment $\alpha(\underline{s})(R - \psi - \mathcal{P})$ can be promised after bad news.

4.3 Optimal asset transfers after bad news

To complete the analysis of the second best, it now only remains to characterise asset transfers. To do so, we build on the above analysis and obtain our next proposition:

Proposition 3 *If*

$$\frac{u'(c_B(\underline{s}))}{u'(c_B(\bar{s}))}\Big|_{\alpha(\underline{s})=0} > \frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - \psi_I(0)}, \quad (21)$$

then, in the second best, the asset transfer is interior, $\alpha(\underline{s}) \in (0, 1)$, and such that

$$\frac{u'(c_B(\underline{s}))}{u'(c_B(\bar{s}))} = \frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - (\psi_I(\alpha(\underline{s})) + \alpha(\underline{s})\psi'_I(\alpha(\underline{s})))}, \quad (22)$$

where $c_B(\underline{s})$ and $c_B(\bar{s})$ are as given in Proposition 2. Otherwise, $\alpha(\underline{s}) = 0$.

To interpret the left-hand-sides of (21) and (22), recall that Lemma 3 implies that $c_B(\underline{s}) < c_B(\bar{s})$. Thus the marginal rate of substitution, $\frac{u'(c_B(\underline{s}))}{u'(c_B(\bar{s}))}$, is larger than 1. This reflects signal-risk induced imperfect insurance. The worse the insurance, the larger this marginal rate of substitution. Thus, the left-hand-sides of (22) and (21) reflect the marginal benefit of an increase in insurance.

While the left-hand-side of (22) and (21) reflect the preferences of the protection buyer, the right-hand-sides of (22) and (21) reflect the technology and incentives of the productive asset holders.

The denominator of the right-hand side of (22) and (21) is the wedge between the productive asset's marginal pledgeable income when it is held by investors ($R - (\psi_I(\alpha(\underline{s})) + \alpha(\underline{s})\psi'_I(\alpha(\underline{s})))$) and its counterpart when it is held by protection sellers (\mathcal{P}). Thus, it measures how much more income you can pledge by transferring the productive asset from the protection seller to the investor. The numerator is a similar wedge between the pledgeable income in the first best ($R - \psi$) and its counterpart under moral hazard (\mathcal{P}) which can be interpreted as a normalisation of the numerator. Thus, the right-hand-sides of (21) and (22) reflect the marginal cost of an increase in insurance.

Thus, condition (21) means that, at $\alpha(\underline{s}) = 0$, the marginal benefit of a small asset transfers exceeds its marginal cost. Since, $\psi'_I \geq 0$ and $\psi''_I \geq 0$, the marginal cost of effort for investors $\psi_I(\alpha(\underline{s})) + \alpha(\underline{s})\psi'_I(\alpha(\underline{s}))$ is increasing. So the right-hand-side of (22) is increasing, and takes its minimum value at $\alpha(\underline{s}) = 0$, as in the right-hand-side of (21). Furthermore, by (1) there exists threshold $\hat{\alpha} < 1$ at which the right-hand-side of (22) goes to infinity. Hence, under (21), there exists an interior value of $\alpha(\underline{s}) \in (0, \hat{\alpha})$ for which the marginal benefit of additional insurance is equal to its marginal cost. This pins down the optimal asset transfer in the second best.

4.4 Power utility

To illustrate the above results, assume

$$u(x) = v(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad (23)$$

and

$$\psi_I = \psi + \delta_0 + \delta_1\alpha, \quad (24)$$

and denote the Pareto weight of investors by $\omega > 0$ and that of protection buyers by $1 - \omega$.

We obtain the following corollary:

Corollary 1 *Under (23) and (24) the optimal asset transfer in the second best is $\alpha(\underline{s})$ such that*

$$\frac{1 + E[\theta|\bar{s}] - \frac{Pr[\underline{s}]}{Pr[\bar{s}]}[\alpha(\underline{s})(R - \psi) + (1 - \alpha(\underline{s}))\mathcal{P}]}{1 + E[\theta|\bar{s}] + \alpha(\underline{s})R + (1 - \alpha(\underline{s}))\mathcal{P}} = \left(\frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - (\psi + \delta_0 + 2\delta_1\alpha(\underline{s}))} \right)^{\frac{1}{\gamma}}, \quad (25)$$

and the consumption of the protection seller is

$$c_B(\bar{s}) = \frac{1 + E[\theta|\bar{s}] - \frac{Pr[\underline{s}]}{Pr[\bar{s}]}[\alpha(\underline{s})(R - \psi) + (1 - \alpha(\underline{s}))\mathcal{P}]}{1 + \left(\frac{\omega}{1-\omega}\right)^{\frac{1}{\gamma}}}, \quad (26)$$

while its counterpart after bad news is

$$c_B(\underline{s}) = \frac{1 + E[\theta|\underline{s}] + \alpha(\underline{s})R + (1 - \alpha(\underline{s}))\mathcal{P}}{1 + \left(\frac{\omega}{1-\omega}\right)^{\frac{1}{\gamma}}}, \quad (27)$$

if the solutions of (25), (26), and (27) are such that (3), (4) and (21) hold.

With power utility, the protection buyers' share of the total consumption of protection buyers and investors is

$$\frac{1}{1 + \left(\frac{\omega}{1-\omega}\right)^{\frac{1}{\gamma}}},$$

after both signals. Moreover, asset transfers are independent from the Pareto weights, i.e., there is a separation between production (asset transfers) and allocation decisions. The former set the level of asset transfers which maximises the sum of the protection buyers' and investors' consumptions, independently of ω . The latter allocates that total consumption as a function of the Pareto weight of the investors, ω .

5 Market equilibrium

We now turn from second best to market equilibrium. We first consider complete markets and study whether market equilibrium is information-constrained efficient. Then we analyse the incomplete market case.

The market is complete when a full set of state-contingent securities or contracts can be traded. The publicly observable events upon which securities, contracts and transaction plans can be contingent are the realisations of the signal (\bar{s} or \underline{s} at time 1), the protection buyers asset value ($\bar{\theta}$ or $\underline{\theta}$ at time 2), and the protection sellers' outputs (R or 0 at time 2). While a richer set of markets is conceivable, only the following three markets will be needed:

Market for insurance against the realization of $\tilde{\theta}$: Protection buyers and protection sellers participate in this market at time 0. In line with our simplifying assumption in

the previous section that the Pareto weight of protection sellers is 0, we assume that, in this market, protection buyers have all the bargaining power (so protection sellers are held to their reservation utility). Each protection buyer is matched with one protection seller, and makes him a take it or leave it offer. The offer includes a set of time 2 transfers, $\tau(\theta, s, R)$, through which the two parties engineer risk sharing.¹⁰ A positive transfer $\tau(\theta, s, R) > 0$ denotes a payment from the seller to the buyer and vice versa.

Market for protection sellers' assets: Since we have seen in the previous section that, after bad news, the second best also involves asset transfers from protection sellers to investors, we allow the protection buyer to request his counterparty to sell a fraction $\alpha_S \geq 0$ of his assets after a bad signal. The sale occurs at time 1, after the realisation of the signal, and before effort is exerted. The price is denoted by p . While supply, α_S , stems from protection sellers, demand α_I stems from investors. All market participants are competitive.

The proceeds $\alpha_S p$ belong to the protection seller but are put on an escrow account, in which they are ring fenced from moral hazard, and can be used to pay the protection buyer at time 2. Thus the request to sell a fraction α_S of the asset can be interpreted as a margin call, and the proceeds $\alpha_S p$ as the margin, deposited in a margin account at the central counterparty clearing house (CCP).

Market for insurance against signal risk: When effort is observable, this market is not needed, as protection sellers can fully insure protection buyers against the risk associated with their endowment $\tilde{\theta}$, and are not exposed to signal risk. In contrast, as shown in the previous section, moral hazard limits the extent to which protection sellers can insure protection buyers, leaving the latter exposed to signal risk. This opens the scope for signal-risk sharing between protection buyers and protection sellers. The corresponding market is held at time 0 and enables participants to exchange consumption after bad signal against consumption

¹⁰Thus, each protection seller in our model sells insurance to only one protection buyer. This avoids the problems arising with non-exclusive contracting, analysed by Acharya and Bisin (2013).

after good signal. Owners of one unit of the contract receive q units of consumption good after a bad signal \underline{s} and pay 1 unit of consumption good after a good signal. We denote protection buyers' demand by x_B and investors' supply by x_I .

Equilibrium: Equilibrium in these three markets consists of transfers $\tau(\theta, s, R)$, prices (p, q) and trades (α_S, α_I) and (x_B, x_I) , such that all participants behave optimally and markets clear: $\alpha_I = \alpha_S$ and $x^d = x^s$. To solve for equilibrium, we first write down the incentive and participation constraints of protection sellers. Second, we characterize investors' demand, α_I and x_I . Third, we analyse contracting between protection buyers and protection sellers. Fourth, we characterize equilibrium in the markets for insurance against signal risk and in the market for protection sellers' assets.

5.1 Protection sellers' incentive and participation constraints

Incentive compatibility: As in the second best, it can be shown that, after good news (\bar{s}) the incentive compatibility condition of protection sellers is slack. After a bad signal, the incentive compatibility condition, under which protection sellers exert effort, is

$$(1 - \alpha_S)(R - \psi) + \alpha_S p - E[\tau(\theta, s, R)|\underline{s}] \geq \mu((1 - \alpha_S)R + \alpha_S p - E[\tau(\theta, s, R)|\underline{s}]) + (1 - \mu)E[\max[\alpha_S p - \tau(\theta, s, 0), 0]|\underline{s}]. \quad (28)$$

The left-hand side of (28) is the expected gain of the protection seller on the equilibrium path: He exerts effort and obtains $R - \psi$ for each of the $1 - \alpha_S$ units of asset kept. In addition, the protection seller owns the proceeds from the asset sale, $\alpha_S p$, deposited in the margin account. Finally, the expected net payment by the protection seller to the protection buyer is $E[\tau(\theta, s, R)|\underline{s}]$.

The right-hand side of (28) is the expected profit of the protection seller if he deviates and does not exert effort. In that case, with probability μ , the protection seller's productive asset

still generates R , and the protection seller's expected gain is the same as on the equilibrium path, except that the cost of effort, $(1 - \alpha_S)\psi$, is not incurred. With probability $(1 - \mu)$, the productive asset held by the protection seller generates no output. In that case, because of limited liability, the protection seller cannot pay more than $\alpha_S p$. Hence his gain is $\max[\alpha_S p - \tau(\theta, s, 0), 0]$.

It is optimal to set $\tau(\theta, s, 0) = \alpha_S p$. This relaxes the incentive constraint, by reducing the right-hand side of (28), and does not affect the rest of the analysis because transfers $\tau(\theta, s, 0)$ only occur off the equilibrium path. The protection sellers' incentive constraint thus reduces to

$$\alpha_S p + (1 - \alpha_S)\mathcal{P} \geq E[\tau(\theta, s)|\underline{s}] \quad (29)$$

where we write $\tau(\theta, s, R) = \tau(\theta, s)$ to simplify the notation. The right-hand side of (29) is how much the protection seller expects to pay the protection buyer, which can be interpreted as the implicit debt of the protection seller. The left-hand side of (29) is how much the protection seller can credibly pledge to pay. This is equal to the sum of i) the proceeds of the asset sale, deposited on the margin account ($\alpha_S p$), which are fully pledgeable, and ii) the pledgeable part (\mathcal{P}) of the output (R) obtained on the $1 - \alpha_S$ units of assets kept by the protection seller.

Participation constraint: A protection seller accepts the contract if it gives him equilibrium expected gains no smaller than his autarky payoff, i.e., if

$$\Pr[\bar{s}](R - \psi) + \Pr[\underline{s}]((1 - \alpha_S)(R - \psi) + \alpha_S p) - E[\tau(\theta, s)] \geq R - \psi. \quad (30)$$

5.2 Investors' optimal trades

When selling x_I units of the insurance contract against signal risk, and buying α_I units of the protection sellers' assets, an investor obtains time 2 consumption equal to $1 + x_I$ after good news and $1 - qx_I + \alpha_I(R - \psi_I(\alpha_I) - p)$ after bad news. Hence, investors' expected

utility at $t = 0$ is

$$\Pr[\bar{s}]v(1 + x_I) + \Pr[\underline{s}]v(1 - qx_I + \alpha_I(R - \psi_I(\alpha_I) - p)). \quad (31)$$

Investors' supply of insurance against signal risk: At time 0, investors choose x_I to maximise (31). The first-order condition is¹¹

$$\Pr[\bar{s}]v'(1 + x_I) = \Pr[\underline{s}]qv'(1 - qx_I + \alpha_I(R - \psi_I(\alpha_I) - p)), \quad (32)$$

which implies that x_I decreases in q .¹² (32) rewrites as

$$q = \frac{\Pr[\bar{s}]}{\Pr[\underline{s}]} \frac{v'(1 + x_I)}{v'(1 - qx_I + \alpha_I(R - \psi_I(\alpha_I) - p))}, \quad (33)$$

which states that the price of insurance against signal risk is equal to the probability-weighted marginal rate of substitution between consumption after good and bad news.

Investors' demand for protection sellers' assets: At time 1, investors choose α_I to maximise (31). When $p \geq R - \psi_I(0)$, the price of the asset is so high that investors' demand is 0. Otherwise, their demand is pinned down by the first order condition:

$$p = R - [\psi_I(\alpha_I) + \alpha_I\psi'_I(\alpha_I)], \quad (34)$$

which states that the price is equal to the marginal valuation of the investor for the asset. Because the marginal cost $\psi_I(\alpha_I) + \alpha_I\psi'_I(\alpha_I)$ is increasing, (34) implies that investors' demand for the asset is decreasing in p .¹³

¹¹The second-order condition $\Pr[\bar{s}]v''(1 + x_I) + q^2\Pr[\underline{s}]v''(1 - qx_I + \alpha_I(R - \psi_I(\alpha_I) - p)) < 0$ holds by the concavity of the utility function.

¹²The left-hand side of (32) is decreasing in x_I , while the right-hand side is increasing in x_I . Their intersection pins down the optimal supply of insurance by investors, x_I . Now, the right-hand side is increasing in q . Thus, an increase in q shifts up the right-hand of (32), which leads to an intersection between the right- and left-hand sides of (32) at a lower value of x_I .

¹³Increasing marginal cost also implies the second order condition holds.

5.3 Contracting between protection buyers and sellers

A protection buyer chooses a privately optimal contract specifying transfers $\tau(\theta, s)$ and a sale of the productive asset α_S , and demands x_B units of the insurance against signal risk. The latter generates positive transfers to the protection buyer after bad news: qx_B , and negative transfers after good news: $-x_B$. Correspondingly, the consumption of the protection buyer at time 2 is

$$\theta + \tau(\theta, \bar{s}) - x_B$$

after good news, and

$$\theta + \tau(\theta, \underline{s}) + qx_B$$

after bad news. Thus the program of the protection buyer is to choose x_B , $\tau(\theta, s)$ (for all $\theta \in \{\underline{\theta}, \bar{\theta}\}$ and $s \in \{\underline{s}, \bar{s}\}$), as well as $\alpha_S \in [0, 1]$, to maximise

$$Pr[\bar{s}]E[u(\theta + \tau(\theta, s) - x_B)|\bar{s}] + Pr[\underline{s}]E[u(\theta + \tau(\theta, s) + qx_B)|\underline{s}] \quad (35)$$

subject to the protection seller's incentive and participation constraints, (29) and (30). The next lemma states protection buyers' consumption under optimal transfers $\tau(\theta, s)$ as a function of the asset sale α_S :

Lemma 4 *In equilibrium, in the privately optimal contract between protection buyers and protection sellers, the protection sellers' participation and incentive constraints bind. Moreover the protection buyers receive full insurance conditional on the signal, i.e., for a given realisation of the signal, their consumption does not depend on the realisation of $\tilde{\theta}$:*

$$c_B(\bar{\theta}, \bar{s}) = c_B(\underline{\theta}, \bar{s}) = E[\theta|\bar{s}] - \frac{Pr[\underline{s}]}{Pr[\bar{s}]}[\alpha_S(R - \psi) + (1 - \alpha_S)\mathcal{P}] - x_B, \quad (36)$$

$$c_B(\bar{\theta}, \underline{s}) = c_B(\underline{\theta}, \underline{s}) = E[\theta|\underline{s}] + \alpha_S p + (1 - \alpha_S)\mathcal{P} + qx_B. \quad (37)$$

Lemma 4 is similar to Lemma 1. Both in the second best and in the market equilibrium, protection buyers are fully insured conditional on the signal, and the economic intuition is the same in the two cases.

The next lemma states what fraction of their assets protection sellers are required to sell after bad news:

Lemma 5 *When*

$$p \leq \mathcal{P} + (R - \psi - \mathcal{P}) \frac{u' \left(E[\theta|\bar{s}] - \frac{Pr[\underline{s}]}{Pr[\bar{s}]} \mathcal{P} - x_B \right)}{u'(E[\theta|\underline{s}] + \mathcal{P} + qx_B)} \quad (38)$$

$\alpha_S = 0$, otherwise α_S is strictly positive and such that

$$\frac{u'(E[\theta|\underline{s}] + \alpha_S p + (1 - \alpha_S) \mathcal{P} + qx_B)}{u' \left(E[\theta|\bar{s}] - \frac{Pr[\underline{s}]}{Pr[\bar{s}]} [\alpha_S (R - \psi) + (1 - \alpha_S) \mathcal{P}] - x_B \right)} = \frac{\lambda_1}{(p - \mathcal{P}) Pr[\underline{s}] \lambda_S} + \frac{\frac{\psi}{1-\mu} - \psi}{p - \mathcal{P}}, \quad (39)$$

where λ_1 is the Lagrange multiplier of the constraint $\alpha_S \leq 1$.

Equation (39) is similar to (22). In both cases, the left-hand side is the protection sellers' marginal rate of substitution between consumption after bad news and after good news and therefore reflects the marginal benefit of an increase in insurance. In (22), the right-hand side involves the ratio

$$\frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - (\psi_I(\alpha(\underline{s})) + \alpha(\underline{s}) \psi'_I(\alpha(\underline{s})))},$$

while, in (39), the corresponding ratio is

$$\frac{\frac{\psi}{1-\mu} - \psi}{p - \mathcal{P}}.$$

In both ratios the numerator is the same, and the denominator reflects how much more income can be pledged for insurance by transferring the productive asset from protection sellers to investors.

5.4 Equilibrium

Equilibrium in the market for insurance against signal risk: Taking the first order condition with respect to x_B in (35), protection buyers' trade in that market is x_B such that

$$q = \frac{\Pr[\bar{s}]u'(\theta + \tau(\theta, \bar{s}) - x_B)}{\Pr[\underline{s}]u'(\theta + \tau(\theta, \underline{s}) + qx_B)}. \quad (40)$$

Since the righthand side of (40) is increasing in x_B , (40) implies x_B is increasing in q , while (33) implies that x_I decreases in q . At equilibrium, q is such that $x_B = x_I$. Combining (33) and (40), we obtain our next proposition:

Proposition 4 *Equilibrium in the market for insurance against signal risk involves price q^* and trading volume x^* such that*

$$q^* = \frac{\Pr[\bar{s}]}{\Pr[\underline{s}]} \frac{v'(1 + x^*)}{v'(1 - q^*x^* + \alpha_I(R - \psi_I(\alpha_I) - p))} = \frac{\Pr[\bar{s}]}{\Pr[\underline{s}]} \frac{u'(\theta + \tau(\theta, \bar{s}) - x^*)}{u'(\theta + \tau(\theta, \underline{s}) + q^*x^*)}. \quad (41)$$

Equation (41) states that, in equilibrium, the marginal rates of substitution between consumption after bad news and after good news is equated among protection buyers and protection sellers, i.e., they share risk optimally, as in the second best (see Proposition 2). Moreover, this marginal rate of substitution (weighted by the probabilities of good and bad news) is equal to the price of insurance against signal risk.

As long as protection buyers are exposed to signal risk,

$$\frac{u'(\theta + \tau(\theta, \bar{s}) - x^*)}{u'(\theta + \tau(\theta, \underline{s}) + qx^*)} < 1,$$

which, combined with (41), implies insurance against signal risk is not actuarially fair: investors who supply protection buyers with insurance against bad news earn profits on average. This, in turn, means that investors' equilibrium supply is strictly positive. So the market for insurance against signal risk is active, i.e., $x^* > 0$. This reflects that, because of moral hazard protection sellers cannot fully insure protection buyers, so the latter demand strictly

positive amount of additional insurance from investors.

Equilibrium in the market for protection sellers' assets: Given equilibrium (q^*, x^*) in the market for insurance against signal risk, equilibrium in the market for protection sellers' assets is defined by a price p^* and a trading volume α^* , such that the market clears, i.e., $\alpha_S(p^*) = \alpha_I(p^*) = \alpha^*$. The next proposition characterises equilibrium in the market for protection sellers' assets:

Proposition 5 *If*

$$\frac{u'(E[\theta|\underline{s}] + \mathcal{P} + qx^*)}{u'(E[\theta|\bar{s}] - \frac{Pr[\underline{s}]}{Pr[\bar{s}]} \mathcal{P} - x^*)} > \frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - \psi_I(0)}, \quad (42)$$

the equilibrium level of asset sales α^ is strictly positive and such that*

$$\frac{u'(E[\theta|\underline{s}] + \alpha^*p + (1 - \alpha^*)\mathcal{P} + qx^*)}{u'(E[\theta|\bar{s}] - \frac{Pr[\underline{s}]}{Pr[\bar{s}]}[\alpha^*(R - \psi) + (1 - \alpha^*)\mathcal{P}] - x^*)} = \frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - (\psi_I(\alpha^*) + \alpha^*\psi'_I(\alpha^*))}, \quad (43)$$

while the market price of protection sellers' assets is:

$$p^* = R - [\psi_I(\alpha^*) + \alpha_I\psi'_I(\alpha^*)]. \quad (44)$$

Otherwise, if (42) does not hold, there are no asset sales in equilibrium, i.e., $\alpha^ = 0$.*

5.5 Equilibrium constrained efficiency

Comparing Lemma 4 and Propositions 4 and 5 to Propositions 2 and 3, we analyse the efficiency of market equilibria.

Proposition 6 *Market equilibrium is information constrained Pareto efficient.*

It is striking that, in spite of moral hazard, equilibrium is constrained efficient, all the more so that the price in the incentive constraint (29) induces pecuniary externalities. Proposition 6 reflects the presence of two countervailing pecuniary externalities:

When one protection buyer demands larger margins, this depresses the price, which tightens the incentive constraint for the others. This negative pecuniary externality tends to increase the amount of signal risk protection buyers must bear.

There is, however, a countervailing, stabilising effect. The decline in the price increases the profits of investors after bad signals. Thus, while a negative signal is a negative shock for protection buyers, it is a positive shock for investors. This opens the scope for risk-sharing gains from trade between investors and protection buyers. When the market is complete, investors and protection buyers fully exploit this risk-sharing opportunity, until their marginal rates of substitution are equalised. At that point, the negative effect of the decline in price on the insurance provided by protection sellers is fully offset by its positive effect on the insurance provided by investors.

5.6 Incomplete markets

We now turn to an incomplete market setting, in which all is as above except that there is no market, at time 0, for insurance against signal risk. That is, we constrain $x_B = x_I = 0$. Proceeding along similar lines as with complete markets, one can show that equilibrium in incomplete markets is as follows: In the privately optimal contract between protection buyers and protection sellers, the protection sellers' participation and incentive constraints bind and the protection buyers receive full insurance conditional on the signal. The consumption of the protection buyer after bad news and after good news is as in Lemma 4, except that x_B is set to 0: Similarly, the condition for asset transfers is the same as (42), except that x is set to 0. When that condition does not hold $\alpha^{IM} = 0$, and when it holds α^{IM} is strictly positive and pinned down by the same equation as (43) with x set to 0,

$$\frac{u'(E[\theta|\underline{s}] + \alpha^{IM}p^{IM} + (1 - \alpha^{IM})\mathcal{P})}{u'\left(E[\theta|\bar{s}] - \frac{\text{Pr}[\underline{s}]}{\text{Pr}[\bar{s}]}[\alpha^{IM}(R - \psi) + (1 - \alpha^{IM})\mathcal{P}]\right)} = \frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - (\psi_I(\alpha^{IM}) + \alpha^{IM}\psi'_I(\alpha^{IM}))}, \quad (45)$$

while the equilibrium price is $p^{IM} = R - [\psi_I(\alpha^{IM}) + \alpha_I \psi'_I(\alpha^{IM})]$ as in (44). Finally, investors' consumption net of cost is equal to $1 + \alpha^{IM}(R - \psi_I(\alpha^{IM}) - p^{IM})$ after bad news and 1 after good news. Building on the above remarks, we can now state our next proposition:

Proposition 7 *Equilibrium with incomplete markets is Pareto dominated by equilibrium with complete markets.*

The proposition reflects that investors' and protection buyers' marginal rates of substitution are equalised when the market is complete, but different when the market is incomplete. Trading insurance against signal risk is what enables protection buyers and investors to equalise their marginal rates of substitution. This is precluded when the market is incomplete. Some gains from trade are therefore left on the table. Hence the incomplete market equilibrium allocation is information constrained Pareto dominated.

What are the consequences of market participants' inability to trade insurance against signal risk at time 0? With complete markets, protection buyers can purchase additional insurance against signal risk from protection sellers. This is impossible with incomplete markets, which tends to increase protection buyers' exposure to signal risk. To make up for that increased risk exposure, protection buyers request larger margin calls from protection sellers, to increase the amount of insurance they can obtain from them. This leads to our next result:

Proposition 8 *Either $\alpha^{IM} = \alpha^* = 0$, so that there are no margin calls irrespective of whether the market is complete or not, or $\alpha^{IM} > 0$ in which case margin calls are larger and the price of protection sellers' assets is lower when markets are incomplete than when they are complete.*

Proposition 8 implies that, due to market incompleteness, margin calls are inefficiently high and price for the sellers' assets inefficiently low. That is, market incompleteness leads to inefficient fire-sales. This illustrates that, in our simple general equilibrium context, there

are interactions between markets: The ability for protection buyers and investors to trade insurance against signal risk in one market reduces the need to sell protection sellers' assets in another market. When the former market does not exist, this depresses prices in the latter.

While fire sales are a symptom of the inefficiency induced by market incompleteness, they are not a necessary condition for inefficiency. Even if $\alpha^{IM} = 0$, market incompleteness prevents the equalisation of the marginal rates of substitution of protection buyers and investors, leading to information-constrained inefficient allocation of risks.

6 Implications

6.1 Positive implications

When bad news (\underline{s}) are publicly observed, this lowers the conditional expectation of the final value of protection buyers' assets (to $E[\theta|\underline{s}]$). In the first best, there is no simultaneous change in the valuation of protection sellers' assets: these assets remain in the hands of their most efficient holders, who exert effort, and value each unit of asset at $R - \psi$. The situation is different with moral hazard, when the contract between protection buyers and protection sellers requests variation margin calls after bad news. In that case, asset sales α lead to depressed prices

$$p = R - (\psi_I(\alpha) + \alpha\psi'_I(\alpha)) < R - \psi.$$

Thus, because of moral hazard, after bad shocks there is endogenous positive correlation between protection buyers' assets' valuation and protection sellers' assets' valuation after bad shocks. This can be interpreted as contagion. Three remarks are in order:

- First, the endogenous correlation generated by moral hazard concerns bad shocks only: After good news, the conditional expectation of the final value of protection buyers' assets goes up to $E(\tilde{\theta}|\bar{s})$, but the valuation of protection sellers' assets stays at $R - \psi$.

In that respect, our theoretical results are consistent with the empirical findings that correlation among financial assets is larger during bear markets (see, e.g., Ang and Chen, 2002).

- Second, fire sales and contagion increase with moral hazard. When protection sellers are subject to severe moral hazard (e.g., if their activities are opaque and difficult for third parties to analyse), this reduces their pledgeable income \mathcal{P} , which increases the reliance on margins, and therefore worsens fire sales. To the extent that lower ratings proxy for lower pledgeable income, this is consistent with the observation in McDonald and Paulson (2015) that thresholds triggering variation margin calls reflect credit ratings. For example, McDonald and Paulson (2015) write (on page 93): “Goldman Sachs had 44 transactions with AIG... The threshold (level of market value change required to trigger a collateral payment) was 4% as long as AIGFP is rated in the AA/Aa category.”
- Third, while fire sales and contagion can take place even if the market is complete, they are amplified by market incompleteness. As stated in Proposition 8, α is larger when the market is incomplete, which depresses the price p more after bad news.

6.2 Normative implications

Variation margin calls are privately optimal responses to moral hazard. Yet, they generate price drops, which can be interpreted as fire sales. In this context, bad news triggering margin calls are negative shocks for protection sellers, who must sell at depressed prices, and also for protection buyers, who obtain only limited insurance against their risky position. The same news, however, are also positive shocks for investors buying assets in a fire sale. This increases the scope for ex-ante risk sharing between protection buyers and investors. If the market is incomplete, however, these risk-sharing gains from trade cannot be reaped and, consequently, equilibrium is information-constrained inefficient

This suggests that regulators and market organisers should facilitate ex-ante insurance against the risk of fire sales. When organising the market for insurance against a particular risk, they should also help develop another market: one for insurance against the events triggering margin calls in the first market. This involves

- creating contracts contingent on the events that trigger variation margin calls,
- facilitating the trading of these contracts, e.g., by establishing CCPs and making sure that the news triggering margin calls is observable and contractible for all,
- and anticipating who will buy in fire sales, and ensure they can participate ex ante in the market for insurance against the risk of fire sales.

7 Conclusion

Our general equilibrium analysis of risk-sharing under moral hazard emphasizes interactions between markets. From a positive point of view, our model implies that, when assets traded in one market are used to comply with margin calls associated with another market, moral hazard generates endogenous correlation between the two markets after bad news. From a normative point of view, our analysis implies that this correlation is excessive when markets are incomplete, but not when they are complete. Market organisers and regulators should therefore facilitate ex-ante contracting among all relevant counterparties, in particular by creating and centrally clearing contracts contingent on the events triggering variation margin calls. This should increase markets' resilience to negative shocks and reduce fire sales.

References

- Acharya, V. and A. Bisin, 2014, “Counterparty Risk Externality: Centralized versus Over-the-counter Markets,” *Journal of Economic Theory* 149, 153–182.
- Ang, A. and J. Chen, 2002, “Asymmetric Correlations of Equity Portfolios,” *Journal of Financial Economics* 63, 443–494.
- Bank for International Settlements, 2010, “The Role of Margin Requirements and Haircuts in Procyclicality,” Committee on the Global Financial System Paper No. 36.
- Bignon, V. and G. Vuillemeys, 2018, “The Failure of a Clearinghouse: Empirical Evidence,” Working paper, HEC.
- Bolton, P. and M. Oehmke, 2015, “Should Derivatives be Privileged in Bankruptcy?,” *Journal of Finance* 70, 2353–2394.
- Cifuentes, R., G. Ferrucci and H. S. Shin, 2005, “Liquidity Risk and Contagion,” *Journal of the European Economic Association* 3, 556–566.
- Davila, E. and A. Korinek, 2017, “Pecuniary Externalities in Economies with Financial Frictions”, *Review of Economic Studies* 85, 352–395.
- Duffie, D., M. Scheicher and G. Vuillemeys, 2015, “Central clearing and collateral demand,” *Journal of Financial Economics*, 237-256.
- Geanakoplos, J. and Polemarchakis, 1986, “Existence, regularity and constrained suboptimality of competitive allocations when the asset market is incomplete,” Cowles Foundation Paper, 652.
- Greenwald, B., and J. Stiglitz, 1986, “Externalities in Economies with Imperfect Information and Incomplete Markets,” *Quarterly Journal of Economics* 101, 229-264.
- Gromb D, and D. Vayanos, 2002, “Equilibrium and Welfare in Markets with Financially Constrained Arbitrageurs,” *Journal of Financial Economics* 66, 361–407.
- Holmstrom, B. and J. Tirole, 1997, “Financial Intermediation, Loanable Funds, and the Real Sector,” *Quarterly Journal of Economics* 112, 663–692.

Kilenthong, W and R. Townsend, 2014, “Segregated security exchanges with ex ante rights to trade: A market-based solution to collateral-constrained externalities,” NBER Working paper 20086.

Kocherlakota, N., 1998, “The effects of moral hazard on asset prices when financial markets are complete,” *Journal of Monetary Economics*, 39-56.

Kuong, J., 2014, “Self-fulfilling Fire Sales: Fragility of Collateralized Short-term Debt Markets,” Working Paper, INSEAD.

Lorenzoni, G., 2008, “Inefficient Credit Booms,” *Review of Economic Studies* 75, 809–833.

Myers, S., 1977, “Determinants of Corporate Borrowing,” *Journal of Financial Economics*, 5 (2), 147-175.

Shleifer, A. and R. Vishny, “Fire Sales in Finance and Macroeconomics.” *Journal of Economic Perspectives* 25, 29–48.

Stiglitz, J., 1982, “The inefficiency of the stock market equilibrium,” *Review of Economic Studies*, 49, 241-261.

Thompson, J.R., 2010, “Counterparty Risk in Financial Contracts: Should the Insured Worry about the Insurer?,” *Quarterly Journal of Economics* 125, 1195–1252.

Tirole, J., 2006, *The Theory of Corporate Finance* (Princeton University Press, Princeton).

Kilenthong, W. and R. Townsend, 2017, “A Market Based Solution for Fire Sales and Other Pecuniary Externalities,” Working Paper, MIT.

Proofs

Proof of Proposition 1: (maybe should be put in supplementary appendix)

The Lagrangian is:

$$\begin{aligned}
L_{FB}(c_B(\theta, s), c_S(\theta, s), c_I(\theta, s), \alpha(s)) &= \omega_B E[u(c_B(\theta, s))] + \omega_I E[v(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))] \\
&\quad + \lambda_B [E[u(c_B(\theta, s))] - E[u(\theta)]] \\
&\quad + \lambda_I [E[v(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))] - v(1)] \\
&\quad + \lambda_S [E[c_S(\theta, s) - (1 - \alpha(s))\psi] - (R - \psi)] \\
&\quad + \sum \lambda(\theta, s) [\theta + 1 + R - (c_B(\theta, s) + c_S(\theta, s) + c_I(\theta, s))] \\
&\quad + \sum_s^{\theta, s} (\lambda_1(s)[1 - \alpha(s)] + \lambda_0(s)[\alpha(s)])
\end{aligned}$$

First-order conditions with respect to $c_B(\theta, s)$, $c_I(\theta, s)$, $c_S(\theta, s)$ and $\alpha(s)$ are

$$(\omega_B + \lambda_B) \Pr[\theta, s] u'(\theta, s) = \lambda(\theta, s), \quad (46)$$

$$(\omega_I + \lambda_I) \Pr[\theta, s] v'(\theta, s) = \lambda(\theta, s), \quad (47)$$

$$\lambda_S \Pr[\theta, s] = \lambda(\theta, s), \quad (48)$$

and

$$-(\omega_I + \lambda_I) \Pr[s] E[v'(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))(\psi_I + \alpha(s)\psi'_I)|s] + \lambda_S \Pr[s] \psi = \lambda_1(s) - \lambda_0(s), \quad (49)$$

respectively (where, in (49), we have used $\Pr[\theta, s] = \Pr[\theta|s]\Pr[s]$). The second-order conditions with respect to $c_B(\theta, s)$, $c_I(\theta, s)$ and $c_S(\theta, s)$ hold because of decreasing marginal

utilities. The second-order condition with respect to α , is:

$$\begin{aligned}
& -(\omega_I + \lambda_I)\Pr[s]E[-v''(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))(\psi_I + \alpha(s)\psi_I')^2 \\
& \quad + v'(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))(2\psi_I' + \alpha(s)\psi_I'')|s] \leq 0,
\end{aligned} \tag{50}$$

which holds since $\psi_I'' \geq 0$ and $v'' < 0$.

(46), (47) and (48) imply

$$(\omega_B + \lambda_B)u'(\theta, s) = (\omega_I + \lambda_I)v'(\theta, s) = \lambda_S \quad \forall(\theta, s) \tag{51}$$

Because neither $\omega_B, \omega_I, \lambda_B, \lambda_I$, nor λ_S depend on the state (θ, s) , (51) implies that the marginal utilities of buyers and investors are constant across states. Hence, $c_B(\theta, s) = c_B$ and $c_I(\theta, s) = c_I$.

The resource constraints bind ($\lambda(\theta, s) > 0$). Suppose not. Because $v', u' > 0$, $\Pr[\theta, s] > 0$, this implies $\omega_B + \lambda_B = 0$ and $\omega_I + \lambda_I = 0$, and hence, $\omega_B = \omega_I = 0$. But because we also have $\omega_S = 0$ (by assumption), the planner's objective would then become zero.

The participation constraint of the sophisticated investors binds ($\lambda_S > 0$). Because $\Pr[\theta, s] > 0$, this is immediate once $\lambda(\theta, s) > 0$.

There is no asset transfer in any state ($\alpha(s) = 0$). Suppose there were, i.e., $\alpha(s) > 0$. Using the second equality in (51), dividing by $\lambda_S\Pr[s]$, and rearranging, the first-order condition with respect to $\alpha(s)$ becomes

$$\psi - \psi_I(\alpha(s)) = \frac{\lambda_1}{\lambda_S\Pr[s]} + \alpha(s)\psi_I'$$

Given that $\lambda_S > 0$, $\psi_I' \geq 0$ and $\psi < \psi_I(\alpha(s))$ when $\alpha(s) > 0$, this is a contradiction: the left-hand side is negative while the right-hand side is weakly positive.

Given constant consumption for buyers and investors, and the binding resource con-

straints, we have

$$c_B + c_I + c_S(\theta, s) = \theta + 1 + R \quad \forall(\theta, s).$$

Using this to substitute for $c_S(\theta, s)$ in the binding participation constraint of investors, together with $\alpha(s) = 0$, we have

$$c_B + c_I = E[\theta] + 1.$$

QED

Proof of Lemma 1:

The Lagrangian of the second best maximisation problem is

$$\begin{aligned} L_{SB}(c_B(\theta, s), c_S(\theta, s), c_I(\theta, s), \alpha(s)) &= \omega_B E[u(c_B(\theta, s))] + \omega_I E[v(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))] \\ &+ \sum_s \lambda_{IC(s)} \left[E[c_S(\theta, s)|s] - \frac{(1 - \alpha(s))\psi}{1 - \mu} \right] \\ &+ \lambda_B [E[u(c_B(\theta, s))] - E[u(\theta)]] \\ &+ \lambda_I [E[v(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))] - v(1)] \\ &+ \lambda_S [E[c_S(\theta, s) - (1 - \alpha(s))\psi] - (R - \psi)] \\ &+ \sum_{\theta, s} \lambda(\theta, s) [\theta + 1 + R - (c_B(\theta, s) + c_S(\theta, s) + c_I(\theta, s))] \\ &+ \sum_s (\lambda_1(s)[1 - \alpha(s)] + \lambda_0(s)[\alpha(s)]). \end{aligned}$$

First-order conditions with respect to $c_B(\theta, s)$ and $c_I(\theta, s)$ are the same as in the first best, (46) and (47), respectively. The first order conditions with respect to $c_S(\theta, s)$ and $\alpha(s)$ are altered, to take into account the incentive compatibility constraint, and write

$$\lambda_{IC(s)} \Pr[\theta|s] + \lambda_S \Pr[\theta, s] = \lambda(\theta, s) \tag{52}$$

and

$$-(\omega_I + \lambda_I)\Pr[s]E[v'(\theta, s)|s](\psi_I(\alpha) + \alpha(s)\psi'_I) + \lambda_{IC(s)}\frac{\psi}{1-\mu} + \lambda_S\Pr[s]\psi = \lambda_1(s) - \lambda_0(s), \quad (53)$$

respectively. The second order conditions are as in the first best.

The first order conditions with respect to $c_B(\theta, s)$ and $c_S(\theta, s)$, (46) and (52) respectively, imply

$$u'(\theta, s) = \frac{1}{\omega_B + \lambda_B} \left(\lambda_{IC(s)}\frac{1}{\Pr[s]} + \lambda_S \right) \quad (54)$$

while the first order conditions with respect to $c_I(\theta, s)$ and $c_S(\theta, s)$, (47) and (52) respectively, imply

$$v'(\theta, s) = \frac{1}{\omega_I + \lambda_I} \left(\lambda_{IC(s)}\frac{1}{\Pr[s]} + \lambda_S \right). \quad (55)$$

Because their right-hand sides are independent of θ , (54) and (55) imply that, for a given realisation of the signal s , the marginal utility of consumption of the protection buyers and investors is the same in $(\bar{\theta}, s)$ and $(\underline{\theta}, s)$.

QED

Proof of Lemma 2:

1) First, we prove that the resource constraints bind ($\lambda(\theta, s) > 0$). Suppose not. Because $v', u' > 0$, $\Pr[\theta, s] > 0$, by (46) and (47), this implies $\omega_B + \lambda_B = 0$ and $\omega_I + \lambda_B = 0$, and hence, $\omega_B = \omega_I = 0$, a contradiction.

2) Second we prove that the participation constraint of the protection seller binds. Suppose not ($\lambda_S = 0$). Then using $\lambda(\theta, s) > 0$ in (52) yields $\lambda_{IC(s)} > 0$ for all (θ, s) , i.e., both IC bind. From the binding IC, we have $E[c_S(\theta, s)|s] = \frac{1-\alpha(s)}{1-\mu}\psi$ and hence,

$$E[c_S(\theta, s)] = \Pr[\bar{s}]\frac{1-\alpha(\bar{s})}{1-\mu}\psi + \Pr[\underline{s}]\frac{1-\alpha(\underline{s})}{1-\mu}\psi = (1 - E[\alpha(s)])\frac{\psi}{1-\mu} \quad (56)$$

Substituting this into the slack PC of the protection seller yields

$$(1 - E[\alpha(s)])\frac{\psi}{1 - \mu} - (1 - E[\alpha(s)])\psi > R - \psi$$

and, after some rearranging,

$$-E[\alpha(s)]\psi\frac{\mu}{1 - \mu} > R - \frac{\psi}{1 - \mu}.$$

which contradicts the assumption that $\mathcal{P} > 0$.

3) Third, we prove that one of the two ICs (or both) must bind. If not, then the first-best allocation would solve the second best problem. Now, with the seller's first-best consumption (8) and $\alpha(s) = 0$, the IC after a bad signal becomes

$$\Pr(\bar{\theta}|\underline{s})(\bar{\theta} - E[\theta] + R) + \Pr(\underline{\theta}|\underline{s})(\underline{\theta} - E[\theta] + R) \geq \frac{\psi}{1 - \mu},$$

i.e.,

$$E[\theta|\underline{s}] - E[\theta] + R \geq \frac{\psi}{1 - \mu},$$

which violates assumption (11).

4) Fourth, we prove that both ICs cannot bind at the same time. Suppose they do. Then we have again (56), which after substituting the binding PC of the sophisticated investor and rearranging yields

$$-E[\alpha(s)]\psi\frac{\mu}{1 - \mu} = R - \frac{\psi}{1 - \mu},$$

which contradicts the assumption that $\mathcal{P} > 0$.

QED

Proof of Lemma 3:

1) First, we prove that when the incentive compatibility condition in state s is slack, then $\alpha(s) = 0$. Suppose not, i.e., $\alpha(s) > 0$ and $\lambda_{IC(s)} = 0$. Then using (47) and (52), (53)

becomes

$$-\lambda_S \Pr[s](\psi_I(\alpha(s)) + \alpha(s)\psi'_I(\alpha(s))) + \lambda_S \Pr[s]\psi = \lambda_1(s).$$

Dividing by $\lambda_S \Pr[s] > 0$ and rearranging yields

$$\psi - \psi_I(\alpha(s)) = \frac{\lambda_1(s)}{\lambda_S(s)\Pr[s]} + \alpha(s)\psi'_I(\alpha(s)).$$

Given that $\psi'_I \geq 0$ and $\psi < \psi_I$ when $\alpha(s) > 0$, we obtain the desired contradiction. The left-hand side is negative while the right-hand side is weakly positive.

2) Second, we prove that the incentive compatibility condition after a bad signal binds. Suppose not ($\lambda_{IC(\underline{s})} = 0$) and only the IC after the good signal binds. Now, given that the IC after a bad signal is slack, so that $\alpha(\underline{s}) = 0$, and the IC after a good signal binds, we have

$$\begin{aligned} E[c_S(\theta, s)|\bar{s}] &= \frac{(1 - \alpha(\bar{s}))\psi}{1 - \mu} = \frac{\psi}{1 - \mu} - \frac{\alpha(\bar{s})\psi}{1 - \mu} \\ E[c_S(\theta, s)|\underline{s}] &> \frac{\psi}{1 - \mu} \end{aligned}$$

which implies that

$$E[c_S(\theta, s)|\underline{s}] - E[c_S(\theta, s)|\bar{s}] > 0 \tag{57}$$

Next, from the binding resource constraints and full risk sharing conditional on the signal, we have

$$c_S(\theta, s) = \theta + 1 + R - (c_B(s) + c_I(s)) \tag{58}$$

and hence,

$$E[c_S(\theta, s)|\bar{s}] = E[\theta|\bar{s}] + 1 + R - (c_B(\bar{s}) + c_I(\bar{s})) \tag{59}$$

$$E[c_S(\theta, s)|\underline{s}] = E[\theta|\underline{s}] + 1 + R - (c_B(\underline{s}) + c_I(\underline{s})) \tag{60}$$

so that

$$E[c_S(\theta, s)|\underline{s}] - E[c_S(\theta, s)|\bar{s}] = E[\theta|\underline{s}] - E[\theta|\bar{s}] - [c_B(\underline{s}) - c_B(\bar{s}) + c_I(\underline{s}) - c_I(\bar{s})]. \quad (61)$$

To obtain that expression in (61) is weakly negative, so that we have the contradiction to (57), the term in squared brackets with the consumptions must be weakly positive (because the signal is (weakly) informative, we have $E[\theta|\underline{s}] - E[\theta|\bar{s}] \leq 0$). From (46), (52) and the slack IC after a bad signal,

$$\begin{aligned} (\omega_B + \lambda_B)u'(\theta, \bar{s}) &= \lambda_S + \frac{\lambda_{IC(\bar{s})}}{\Pr[\bar{s}]} \\ (\omega_B + \lambda_B)u'(\theta, \underline{s}) &= \lambda_S \end{aligned}$$

Together with full risk sharing conditional on the signal, this implies that

$$c_B(\underline{s}) \geq c_B(\bar{s}).$$

The same type of argument also establishes that

$$c_I(\underline{s}) \geq c_I(\bar{s}).$$

Hence, the term in in squared brackets in (61) is (weakly) positive, which yields the desired contradiction.

3) Third, we analyse the ranking of the consumptions of the protection buyers after bad and good signals. Combining (47) with (52), and using the fact that there is full risk sharing conditional on the signal and that only the IC after the bad signal binds, we obtain:

$$\begin{aligned} (\omega_B + \lambda_B)\Pr[\theta, \bar{s}]u'(c_B(\bar{s})) &= \lambda_S\Pr[\theta, \bar{s}] \\ (\omega_B + \lambda_B)\Pr[\theta, \underline{s}]u'(c_B(\underline{s})) &= \lambda_{IC(\underline{s})}\Pr[\theta|\underline{s}] + \lambda_S\Pr[\theta, \underline{s}] \end{aligned}$$

so that

$$\frac{u'(c_B(\underline{s}))}{u'(c_B(\bar{s}))} = 1 + \frac{\lambda_{IC(\underline{s})}}{\Pr[\underline{s}]\lambda_S} \quad (62)$$

Because $\lambda_{IC(\underline{s})} > 0$ and $\lambda_S > 0$, we have imperfect risk sharing across signals with

$$c_B(\underline{s}) < c_B(\bar{s}).$$

QED

Proof of Proposition 3:

1) First, we write down more precisely the first order optimality condition with respect to $\alpha(\underline{s})$. Using (47) and Lemma 1, the derivative of the Lagrangian with respect to $\alpha(\underline{s})$ is

$$\frac{\partial L_{SB}}{\partial \alpha(\underline{s})} = -\lambda(\theta, s)\Pr[\underline{s}]\frac{1}{\Pr[\theta, \underline{s}]}(\psi_I(\alpha(\underline{s})) + \alpha(\underline{s})\psi'_I(\alpha(\underline{s}))) + \lambda_{IC(\underline{s})}\frac{\psi}{1-\mu} + \lambda_S\Pr[\underline{s}]\psi - (\lambda_1(\underline{s}) - \lambda_0(\underline{s})).$$

Using (52), this rewrites

$$\frac{\partial L_{SB}}{\partial \alpha(\underline{s})} = -(\lambda_S\Pr[\underline{s}] + \lambda_{IC(\underline{s})})(\psi_I(\alpha(\underline{s})) + \alpha(\underline{s})\psi'_I(\alpha(\underline{s}))) + \lambda_{IC(\underline{s})}\frac{\psi}{1-\mu} + \lambda_S\Pr[\underline{s}]\psi - (\lambda_1(\underline{s}) - \lambda_0(\underline{s})).$$

Collecting terms

$$\begin{aligned} \frac{\partial L_{SB}}{\partial \alpha(\underline{s})} &= \lambda_{IC(\underline{s})} \left[\frac{\psi}{1-\mu} - (\psi_I(\alpha(\underline{s})) + \alpha(\underline{s})\psi'_I(\alpha(\underline{s}))) \right] \\ &\quad + \lambda_S\Pr[\underline{s}] [\psi - (\psi_I(\alpha(\underline{s})) + \alpha(\underline{s})\psi'_I(\alpha(\underline{s}))) - (\lambda_1(\underline{s}) - \lambda_0(\underline{s}))]. \end{aligned} \quad (63)$$

2) Second, we show that, under (21), there must be some asset transfer, i.e., $\alpha(\underline{s}) > 0$. Suppose not, i.e., suppose we have $\alpha(\underline{s}) = 0$, then $\lambda_1(\underline{s}) = 0$ and, by (63), the optimality condition, such that $\alpha(\underline{s}) = 0$, $\frac{\partial L_{SB}}{\partial \alpha(\underline{s})} \leq 0$ writes as

$$\frac{\lambda_{IC(\underline{s})}}{\lambda_S\Pr[\underline{s}]} \left[\frac{\psi}{1-\mu} - \psi_I(0) \right] + [\psi - \psi_I(0)] \leq -\frac{\lambda_0(\underline{s})}{\lambda_S\Pr[\underline{s}]}. \quad (64)$$

Now, (62) yields

$$\frac{\lambda_{IC}(\underline{s})}{\Pr[\underline{s}]\lambda_S} = \frac{u'(c_B(\underline{s}))}{u'(c_B(\bar{s}))}\Big|_{\alpha(\underline{s})=0} - 1. \quad (65)$$

Substituting into (64) yields

$$\frac{u'(c_B(\underline{s}))}{u'(c_B(\bar{s}))}\Big|_{\alpha(\underline{s})=0} - \frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - \psi_I(0)} \leq -\frac{\lambda_0(\underline{s})}{\lambda_S \Pr[\underline{s}] \left[\frac{\psi}{1-\mu} - \psi_I(0) \right]}. \quad (66)$$

which contradicts (21), since the latter states that

$$\frac{u'(c_B(\underline{s}))}{u'(c_B(\bar{s}))}\Big|_{\alpha(\underline{s})=0} > \frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - \psi_I(0)}.$$

4) Third, we characterise asset transfers when they are interior, i.e., when $\alpha(\underline{s}) \in (0, 1)$.

In that case, (63) and (62) imply

$$\left[\frac{u'(c_B(\underline{s}))}{u'(c_B(\bar{s}))} - 1 \right] + \frac{\psi - (\psi_I(\alpha(\underline{s})) + \alpha(\underline{s})\psi'_I(\alpha(\underline{s})))}{\frac{\psi}{1-\mu} - (\psi_I(\alpha(\underline{s})) + \alpha(\underline{s})\psi'_I(\alpha(\underline{s})))} = 0.$$

That is

$$\frac{u'(c_B(\underline{s}))}{u'(c_B(\bar{s}))} = \frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - (\psi_I(\alpha(\underline{s})) + \alpha(\underline{s})\psi'_I(\alpha(\underline{s})))},$$

where $c_B(\underline{s})$ and $c_B(\bar{s})$ are as given in Proposition 2.

QED

Proof of Corollary 1:

After good news, condition (17) writes

$$\left(\frac{c_I(\bar{s})}{c_B(\bar{s})} \right)^\gamma = \frac{\omega}{1-\omega}.$$

That is

$$c_I(\bar{s}) = \left(\frac{\omega}{1-\omega} \right)^{\frac{1}{\gamma}} c_B(\bar{s}).$$

Correspondingly

$$c_I(\bar{s}) + c_B(\bar{s}) = \left[1 + \left(\frac{\omega}{1-\omega} \right)^{\frac{1}{\gamma}} \right] c_B(\bar{s}).$$

Substituting into (15) we have (26).

Similarly, after bad news, condition (17) yields

$$c_I(\underline{s}) + c_B(\underline{s}) = \left[1 + \left(\frac{\omega}{1-\omega} \right)^{\frac{1}{\gamma}} \right] c_B(\underline{s}).$$

Substituting into (14) we have (27).

Moreover, in our example, condition (22) writes:

$$\frac{c_B(\bar{s})}{c_B(\underline{s})} = \left(\frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - (\psi + \delta_0 + 2\delta_1\alpha(\underline{s}))} \right)^{\frac{1}{\gamma}}.$$

Substituting (26) and (27) we obtain the equation for $\alpha(\underline{s})$:

$$\frac{1 + E[\theta|\bar{s}] - \frac{\Pr[\bar{s}]}{\Pr[\underline{s}]}[\alpha(\underline{s})(R - \psi) + (1 - \alpha(\underline{s}))\mathcal{P}]}{1 + E[\theta|\underline{s}] + \alpha(\underline{s})R + (1 - \alpha(\underline{s}))\mathcal{P}} = \left(\frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - (\psi + \delta_0 + 2\delta_1\alpha(\underline{s}))} \right)^{\frac{1}{\gamma}}.$$

QED

Proof of Lemma 4: 1) First, we write down the Lagrangian of the protection buyer and use it to show that the participation constraint of protection sellers bind. The Lagrangian is:

$$\begin{aligned} L(\tau(\theta, s), \alpha_S, x_B) &= \Pr[\bar{s}]E[u(\theta + \tau(\theta, s) - x_B)|\bar{s}] + \Pr[\underline{s}]E[u(\theta + \tau(\theta, s) + qx_B)|\underline{s}] & (67) \\ &+ \lambda_{IC} [\alpha_S p + (1 - \alpha_S)\mathcal{P} - E[\tau(\theta, s)|\underline{s}]] \\ &+ \lambda_S [\Pr[\bar{s}](R - \psi) + \Pr[\underline{s}]((1 - \alpha_S)(R - \psi) + \alpha_S p) - E[\tau(\theta, s)] - (R - \psi)] \\ &+ \lambda_1 [1 - \alpha_S] - \lambda_0 \alpha_S. \end{aligned}$$

The first-order conditions of (67) with respect to $\tau(\theta, \bar{s})$ and $\tau(\theta, \underline{s})$ are:

$$\Pr[\bar{s}]\Pr[\theta|\bar{s}]u'(\theta, \bar{s}) = \lambda_S\Pr[\theta, \bar{s}] \quad \forall\theta$$

$$\Pr[\underline{s}]\Pr[\theta|\underline{s}]u'(\theta, \underline{s}) = \lambda_S\Pr[\theta, \underline{s}] + \lambda_{IC}\Pr[\theta|\underline{s}] \quad \forall\theta$$

which simplify to

$$u'(\theta, \bar{s}) = \lambda_S \quad \forall\theta \quad (68)$$

$$u'(\theta, \underline{s}) = \lambda_S + \frac{\lambda_{IC}}{\Pr[\underline{s}]} \quad \forall\theta. \quad (69)$$

(68) implies that $\lambda_S > 0$, i.e., the participation constraint of protection sellers binds.

2) Second, we use the first order conditions with respect to $\tau(\theta, \bar{s})$ and $\tau(\theta, \underline{s})$ to show that the protection buyer is fully insured conditional on the signal. Because the right-hand side of (68) and (69) do not depend on θ , we have $u'(\theta, \bar{s}) = u'(\theta, \underline{s})$, $\forall\theta$, i.e.,

$$\bar{\theta} + \tau(\bar{\theta}, \bar{s}) = \underline{\theta} + \tau(\underline{\theta}, \bar{s}) \quad (70)$$

$$\bar{\theta} + \tau(\bar{\theta}, \underline{s}) = \underline{\theta} + \tau(\underline{\theta}, \underline{s}). \quad (71)$$

Thus, conditional on the realization of the signal s , the protection buyer is fully insured against remaining θ -risk.

3) Third we prove by contradiction that the incentive compatibility condition of the protection seller binds. To do so, we proceed in two steps.

The first step is to prove that, if the incentive compatibility condition of the protection seller was slack, there would be no asset sale in equilibrium. This first step proceeds by contradiction. Suppose $\lambda_{IC} = 0$ and $\alpha_S = \alpha_I > 0$. Consider the first-order condition of the Lagrangian (67) with respect to α_S , when $\alpha_S > 0$ (and hence, $\lambda_0 = 0$) and $\lambda_{IC} = 0$:

$$-\lambda_s\Pr[\underline{s}](R - \psi - p^*) = \lambda_1. \quad (72)$$

where p^* is the equilibrium price in the asset market. From the investors' demand for the productive asset, we know that $\alpha_I > 0$ requires $p^* < R - \psi_I(0)$. Because $\psi_I(0) \geq \psi$ by assumption, the left-hand side of (72) is strictly negative, which contradicts the fact that the right-hand side is weakly positive.

The second step is to prove that slack protection seller's IC would contradict our assumption that $\mathcal{P} < E[\tilde{\theta}] - E[\tilde{\theta}|\underline{s}]$. Suppose $\lambda_{IC} = 0$ (68) and (69) imply full insurance, $\tau(\theta, \bar{s}) = \tau(\theta, \underline{s}) \equiv \tau(\theta)$ for all θ , and $\bar{\theta} + \tau(\bar{\theta}) = \underline{\theta} + \tau(\underline{\theta})$. Using that, when $\lambda_{IC} = 0$, $\alpha_S = 0$ and there is full insurance, and substituting the binding participation constraint, we obtain $\tau(\bar{\theta}) = -(1 - \pi)(\bar{\theta} - \underline{\theta})$ and $\tau(\underline{\theta}) = \pi(\bar{\theta} - \underline{\theta})$. Using this in the slack incentive constraint yields

$$\begin{aligned} \mathcal{P} &> \Pr[\bar{\theta}|\underline{s}](-1)(1 - \pi)(\bar{\theta} - \underline{\theta}) + (1 - \Pr[\bar{\theta}|\underline{s}])\pi(\bar{\theta} - \underline{\theta}) \\ &= (\pi - \Pr[\bar{\theta}|\underline{s}])(\bar{\theta} - \underline{\theta}) \\ &= E[\theta] - E[\theta|\underline{s}]. \end{aligned}$$

4) Fourth, we compute the transfers. The binding incentive and participation constraints imply

$$E[\tau(\theta, s)|\bar{s}] = -\frac{\Pr[\underline{s}]}{\Pr[\bar{s}]}[\alpha_S(R - \psi) + (1 - \alpha_S)\mathcal{P}], \quad (73)$$

$$E[\tau(\theta, s)|\underline{s}] = \alpha_S p + (1 - \alpha_S)\mathcal{P}. \quad (74)$$

Equations (73) and (74), together with full insurance conditional on the signal, (70) and (71), yield the set of transfers, for a given α_S :

$$\begin{aligned}
\tau^*(\bar{\theta}, \underline{s}) &= -Pr[\underline{\theta}|\underline{s}](\bar{\theta} - \underline{\theta}) + \alpha_S p + (1 - \alpha_S)\mathcal{P}, \\
\tau^*(\underline{\theta}, \underline{s}) &= Pr[\bar{\theta}|\underline{s}](\bar{\theta} - \underline{\theta}) + \alpha_S p + (1 - \alpha_S)\mathcal{P}, \\
\tau^*(\bar{\theta}, \bar{s}) &= -Pr[\underline{\theta}|\bar{s}](\bar{\theta} - \underline{\theta}) - \frac{Pr[\underline{s}]}{Pr[\bar{s}]}[\alpha_S(R - \psi) + (1 - \alpha_S)\mathcal{P}], \\
\tau^*(\underline{\theta}, \bar{s}) &= Pr[\bar{\theta}|\bar{s}](\bar{\theta} - \underline{\theta}) - \frac{Pr[\underline{s}]}{Pr[\bar{s}]}[\alpha_S(R - \psi) + (1 - \alpha_S)\mathcal{P}]
\end{aligned}$$

QED

Proof of Lemma 5: The first-order condition of the Lagrangian (67) with respect to α_S is

$$\lambda_{IC}(p - \mathcal{P}) - \lambda_s Pr[\underline{s}](R - \psi - p) = \lambda_1 - \lambda_0 \quad (75)$$

From (68) and (69) we have

$$\frac{u'(\theta, \underline{s})}{u'(\theta, \bar{s})} = 1 + \frac{\lambda_{IC}}{Pr[\underline{s}]\lambda_S} > 1, \quad (76)$$

where the inequality follows from the binding IC stated in Lemma 4.

Combining (75) and (76), and using the consumptions in Lemma 4, we obtain

$$\frac{u'(E[\theta|\underline{s}] + \alpha_S p + (1 - \alpha_S)\mathcal{P} + qx^d)}{u'\left(E[\theta|\bar{s}] - \frac{Pr[\underline{s}]}{Pr[\bar{s}]}[\alpha_S(R - \psi) + (1 - \alpha_S)\mathcal{P}] - x^d\right)} = \frac{\lambda_1 - \lambda_0}{(p - \mathcal{P})Pr[\underline{s}]\lambda_S} + \frac{R - \psi - \mathcal{P}}{p - \mathcal{P}}. \quad (77)$$

Next, we show that when $p > \mathcal{P} + (R - \psi - \mathcal{P})\frac{u'(E[\theta|\bar{s}] - \frac{Pr[\underline{s}]}{Pr[\bar{s}]}[\mathcal{P} - x^d])}{u'(E[\theta|\underline{s}] + \mathcal{P} + qx^d)}$ then $\alpha_S > 0$. In that case, (77) with $\lambda_0 = 0$ yields (39). Suppose not, $\alpha_S = 0$, so that $\lambda_0 > 0$ and $\lambda_1 = 0$. Then solving (77) with $\alpha_S = 0$ for p yields

$$\left[\frac{u'(E[\theta|\underline{s}] + \mathcal{P} + qx^d)}{u'\left(E[\theta|\bar{s}] - \frac{Pr[\underline{s}]}{Pr[\bar{s}]}[\mathcal{P} - x^d]\right)} + \frac{\lambda_0}{(p - \mathcal{P})Pr[\underline{s}]\lambda_S} \right] (p - \mathcal{P}) = R - \psi - \mathcal{P}$$

$$p = \mathcal{P} + \frac{R - \psi - \mathcal{P}}{\left[\frac{u'(E[\theta|\underline{s}] + \mathcal{P} + qx^d)}{u'(E[\theta|\bar{s}] - \frac{\text{Pr}[\underline{s}]}{\text{Pr}[\bar{s}]} \mathcal{P} - x^d)} + \frac{\lambda_0}{(p - \mathcal{P})\text{Pr}[\underline{s}]\lambda_S} \right]}.$$

This contradicts the assumption that $p > \mathcal{P} + (R - \psi - \mathcal{P}) \frac{u'(E[\theta|\bar{s}] - \frac{\text{Pr}[\underline{s}]}{\text{Pr}[\bar{s}]} \mathcal{P} - x^d)}{u'(E[\theta|\underline{s}] + \mathcal{P} + qx^d)}$, because

$$\frac{R - \psi - \mathcal{P}}{\left[\frac{u'(E[\theta|\underline{s}] + \mathcal{P} + qx^d)}{u'(E[\theta|\bar{s}] - \frac{\text{Pr}[\underline{s}]}{\text{Pr}[\bar{s}]} \mathcal{P} - x^d)} + \frac{\lambda_0}{(p - \mathcal{P})\text{Pr}[\underline{s}]\lambda_S} \right]} < (R - \psi - \mathcal{P}) \frac{u'(E[\theta|\bar{s}] - \frac{\text{Pr}[\underline{s}]}{\text{Pr}[\bar{s}]} \mathcal{P} - x^d)}{u'(E[\theta|\underline{s}] + \mathcal{P} + qx^d)},$$

as

$$1 < \frac{u'(E[\theta|\bar{s}] - \frac{\text{Pr}[\underline{s}]}{\text{Pr}[\bar{s}]} \mathcal{P} - x^d)}{u'(E[\theta|\underline{s}] + \mathcal{P} + qx^d)} \left[\frac{u'(E[\theta|\underline{s}] + \mathcal{P} + qx^d)}{u'(E[\theta|\bar{s}] - \frac{\text{Pr}[\underline{s}]}{\text{Pr}[\bar{s}]} \mathcal{P} - x^d)} + \frac{\lambda_0}{(p - \mathcal{P})\text{Pr}[\underline{s}]\lambda_S} \right],$$

due to

$$1 < 1 + \frac{\lambda_0}{(p - \mathcal{P})\text{Pr}[\underline{s}]\lambda_S} \frac{u'(E[\theta|\bar{s}] - \frac{\text{Pr}[\underline{s}]}{\text{Pr}[\bar{s}]} \mathcal{P} - x^d)}{u'(E[\theta|\underline{s}] + \mathcal{P} + qx^d)}.$$

Finally we show that when $p \leq \mathcal{P} + (R - \psi - \mathcal{P}) \frac{u'(E[\theta|\bar{s}] - \frac{\text{Pr}[\underline{s}]}{\text{Pr}[\bar{s}]} \mathcal{P} - x^d)}{u'(E[\theta|\underline{s}] + \mathcal{P} + qx^d)}$, then $\alpha_S = 0$. To do so we proceed in three steps, corresponding to different values of p .

First, when $p < \mathcal{P}$ then $\alpha_S = 0$. Suppose not, $\alpha_S > 0$ and hence $\lambda_0 = 0$. Then the first term on the right-hand side of (77) is weakly negative and the second term is strictly negative. Hence, the right-hand side is strictly negative while the left-hand side is strictly positive.

Second, when $\mathcal{P} < p \leq \mathcal{P} + (R - \psi - \mathcal{P}) \frac{u'(E[\theta|\bar{s}] - \frac{\text{Pr}[\underline{s}]}{\text{Pr}[\bar{s}]} \mathcal{P} - x^d)}{u'(E[\theta|\underline{s}] + \mathcal{P} + qx^d)}$ then $\alpha_S = 0$. Suppose not, $\alpha_S > 0$ and hence, $\lambda_0 = 0$. Then solving (77) for p yields

$$p = \frac{\lambda_1}{\text{Pr}[\underline{s}]\lambda_S} \frac{u'(E[\theta|\bar{s}] - \frac{\text{Pr}[\underline{s}]}{\text{Pr}[\bar{s}]} [\alpha_S(R - \psi) + (1 - \alpha_S)\mathcal{P}] - x^d)}{u'(E[\theta|\underline{s}] + \alpha_S p + (1 - \alpha_S)\mathcal{P} + qx^d)} + \mathcal{P} + (R - \psi - \mathcal{P}) \frac{u'(E[\theta|\bar{s}] - \frac{\text{Pr}[\underline{s}]}{\text{Pr}[\bar{s}]} [\alpha_S(R - \psi) + (1 - \alpha_S)\mathcal{P}] - x^d)}{u'(E[\theta|\underline{s}] + \alpha_S p + (1 - \alpha_S)\mathcal{P} + qx^d)}.$$

This price decreases when α_S decreases (since the ratio of marginal utilities is strictly increasing in α_S). Yet, with $\alpha_S > 0$, the price will always be larger than the largest price

allowed in the starting condition

$$p = \mathcal{P} + (R - \psi - \mathcal{P}) \frac{u'(E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]} \mathcal{P} - x^d)}{u'(E[\theta|\underline{s}] + \mathcal{P} + qx^d)}$$

because $\lambda_1 \geq 0$ and

$$\frac{u'(E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]} [\alpha_S(R - \psi) + (1 - \alpha_S)\mathcal{P}] - x^d)}{u'(E[\theta|\underline{s}] + \alpha_S p + (1 - \alpha_S)\mathcal{P} + qx^d)} > \frac{u'(E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]} \mathcal{P} - x^d)}{u'(E[\theta|\underline{s}] + \mathcal{P} + qx^d)}$$

when $\alpha_S > 0$.

Third, when $p = \mathcal{P}$, then $\alpha_S = 0$. Suppose not, $\alpha_S > 0$ and hence, $\lambda_0 = 0$. As $p \rightarrow \mathcal{P}$, the right-hand side of (77) goes to infinity, contradiction since the left-hand side is finite.

QED

Proof of Proposition 5:

Lemma 5 states that if

$$p \leq \mathcal{P} + (R - \psi - \mathcal{P}) \frac{u'(E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]} \mathcal{P} - x_B)}{u'(E[\theta|\underline{s}] + \mathcal{P} + qx_B)}$$

then $\alpha_S = 0$, otherwise $\alpha_S > 0$, where is given by (38).

Moreover, the above analysis of investors trades showed that if $p < R - \psi_I(0)$ then $\alpha_I > 0$, while otherwise $\alpha_I = 0$. So two cases must be distinguished.

If

$$\frac{u'(E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]} \mathcal{P} - x_B)}{u'(E[\theta|\underline{s}] + \mathcal{P} + qx_B)} > \frac{\frac{\psi}{1-\mu} - \psi}{R - \psi - \mathcal{P}},$$

then $\alpha^* = 0$ and p^* is any price in $[R - \psi_I(0), \hat{p}(x^d, q)]$.

Otherwise, there exists (p^*, α^*) such that $\alpha_S(p^*) = \alpha_I(p^*) = \alpha^* > 0$. A sufficient condition for $\alpha^* < 1$ is provided by (1), which implies $\psi_I(1) + \psi'_I > \frac{\psi}{1-\mu}$. To see this proceed by contradiction and suppose $\alpha^* = 1$. Then, (34) implies the price is $p^* = R - (\psi_I(1) + \psi'_I)$.

Substituting into (39)

$$\frac{u'(E[\theta|\underline{s}] + \alpha_S p + (1 - \alpha_S)\mathcal{P} + qx^d)}{u'\left(E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]}[\alpha_S(R - \psi) + (1 - \alpha_S)\mathcal{P}] - x^d\right)} = \frac{\lambda_1}{\left(\frac{\psi}{1-\mu} - (\psi_I(1) + \psi'_I)\right) \lambda_S \Pr[\underline{s}] + \frac{R - \psi - \mathcal{P}}{\frac{\psi}{1-\mu} - (\psi_I(1) + \psi'_I)}}$$

The left-hand side is strictly positive, but if $\psi_I(1) + \psi'_I > \frac{\psi}{1-\mu}$ the right-hand side is strictly negative, so we have a contradiction.

In that second case, the price p^* is obtained by applying (34). Substituting this price into (39), while setting $\lambda_1 = 0$, yields (43).

QED

Proof of Proposition 6:

To prove Proposition 6, we first recall the equilibrium conditions, then we recall the second best conditions, and finally we show that for any allocation that satisfies the equilibrium condition, there exists a set of Pareto weights such that this allocation satisfies the condition for second best optimality.

Equilibrium allocation: Substituting equilibrium prices and trades α^*, p^*, x^* , and q^* into (36) and (37) equilibrium protection buyers' consumption is

$$c_B(\bar{\theta}, \bar{s}) = c_B(\underline{\theta}, \bar{s}) = E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]}[\alpha^*(R - \psi) + (1 - \alpha^*)\mathcal{P}] - x^*, \quad (78)$$

$$c_B(\bar{\theta}, \underline{s}) = c_B(\underline{\theta}, \underline{s}) = E[\theta|\underline{s}] + \alpha^* p^* + (1 - \alpha^*)\mathcal{P} + q^* x^*. \quad (79)$$

Similarly substituting α^*, p^*, x^* , and q^* into investors' consumptions

$$c_I(\bar{\theta}, \bar{s}) = c_I(\underline{\theta}, \bar{s}) = 1 + x^*, \quad (80)$$

$$c_I(\bar{\theta}, \underline{s}) = c_I(\underline{\theta}, \underline{s}) = 1 - q^* x^* + \alpha^*(R - p^*). \quad (81)$$

Adding to (78) and (79) to (80) and (81), total consumption of protection buyers and investors is

$$c_B(\theta, \bar{s}) + c_I(\theta, \bar{s}) = 1 + E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]}[\alpha^*(R - \psi) + (1 - \alpha^*)\mathcal{P}], \forall \theta, \quad (82)$$

and

$$c_B(\theta, \underline{s}) + c_I(\theta, \underline{s}) = 1 + E[\theta|\underline{s}] + \alpha^*R + (1 - \alpha^*)\mathcal{P}, \forall \theta. \quad (83)$$

Substituting α^*, p^*, x^*, q^* , (78) and (79) into (41), marginal rates of substitution between consumption after good news and after bad news are equalised for protection buyers and investors.

$$\frac{v'(c_I(\theta, \underline{s}) - \alpha^*\psi_I(\alpha^*))}{v'(c_I(\theta, \bar{s}))} = \frac{u'(c_B(\theta, \underline{s}))}{u'(c_B(\theta, \bar{s}))}. \quad (84)$$

Substituting (78) and (79) into condition (42), the condition writes as

$$\frac{u'(c_B(\underline{s}))}{u'(c_B(\bar{s}))}\Big|_{\alpha=0} > \frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - \psi_I(0)}. \quad (85)$$

When that condition does not hold, $\alpha^* = 0$. When it holds, substituting α^*, p^*, x^*, q^* , into (5), the marginal rate of substitution between consumption after bad news and consumption after good news is equal to what we interpreted, in the discussion of equation (22) in Proposition 3, as the marginal cost of insurance:

$$\frac{u'(c_B(\theta, \underline{s}))}{u'(c_B(\theta, \bar{s}))} = \frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - (\psi_I(\alpha^*) + \alpha^*\psi'_I(\alpha^*))}. \quad (86)$$

Second best allocation: Equations (14) and (15) state the total consumption of protection buyers and investors, after bad news and after good news, in the second best:

$$c_B(\underline{s}) + c_I(\underline{s}) = 1 + E[\theta|\underline{s}] + \alpha(\underline{s})R + (1 - \alpha(\underline{s}))\mathcal{P}, \quad (87)$$

$$c_B(\bar{s}) + c_I(\bar{s}) = 1 + E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]}[\alpha(\underline{s})(R - \psi) + (1 - \alpha(\underline{s}))\mathcal{P}]. \quad (88)$$

Equation (16) states that in the second best marginal rates of substitution are equalised between protection buyers and investors

$$\frac{v'(c_I(\underline{s}) - \alpha(\underline{s})\psi_I(\underline{s}))}{v'(c_I(\bar{s}) - \alpha(\bar{s})\psi_I(\bar{s}))} = \frac{u'(c_B(\underline{s}))}{u'(c_B(\bar{s}))}. \quad (89)$$

Inequality (21) states the condition under which asset transfers are strictly positive in the second best

$$\frac{u'(c_B(\underline{s}))}{u'(c_B(\bar{s}))}\Big|_{\alpha(\underline{s})=0} > \frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - \psi_I(0)}, \quad (90)$$

if that condition does not hold then there are no asset transfers in the second best.

Equation (22)

$$\frac{u'(c_B(\underline{s}))}{u'(c_B(\bar{s}))} = \frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - (\psi_I(\alpha(\underline{s})) + \alpha(\underline{s})\psi'_I(\alpha(\underline{s})))}. \quad (91)$$

Finally, equation (17) which states how total consumption is split between protection buyers and investors as a function of their Pareto weights

$$\frac{u'(c_B(s))}{v'(c_I(s) - \alpha(s)\psi_I(\alpha(s)))} = \frac{\omega_I + \lambda_I}{\omega_B + \lambda_B}. \quad (92)$$

Investors and protection buyers consumptions and asset transfers such that (87), (88), (89), (90), (91) and (92) hold are second best.

Comparing second best and equilibrium allocations: Out of the 6 conditions for second best efficiency, 5 are met in the market equilibrium: (83) is equivalent to (87), (82) is equivalent to (88), (84) is equivalent to (89), (85) is equivalent to (??), and (86) is equivalent to (91). So, market equilibrium is second best iff there exists Pareto weights ω_I and ω_B such that (92) holds for the market equilibrium consumptions. Now, in the market equilibrium, neither the participation constraint of investors, nor that of protection buyers

bind: investors are strictly better off than in autarchy since they strictly prefer to trade in the market for insurance against signal risk, and protection buyers are strictly better off since they can, at least extract all the surplus from contracting with protection sellers with $\alpha = 0$. Consequently, $\lambda_I = \lambda_B = 0$. Hence, (91) holds for the equilibrium consumptions iff there exists Pareto weights ω_I and ω_B such that

$$\frac{u'(c_B(s))}{v'(c_I(s) - \alpha(s)\psi_I(\alpha(s)))} = \frac{\omega_I}{\omega_B}.$$

This is always the case. To see this, pick an arbitrary ω_B , then set

$$\omega_I = \omega_B \frac{u'(c_B(s))}{v'(c_I(s) - \alpha(s)\psi_I(\alpha(s)))}.$$

QED

Proof of Proposition 7:

The marginal rate of substitution of investors is

$$\frac{v'(1 + \alpha^{IM}(R - \psi_I(\alpha^{IM}) - p^{IM}))}{v'(1)}$$

which is always lower than or equal to 1. When $\alpha^{IC} = 0$, the marginal rate of substitution of protection sellers is

$$\frac{u'(E[\theta|\underline{s}] + \mathcal{P})}{u'\left(E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]} \mathcal{P}\right)}$$

which is strictly larger than 1 because of (11). When $\alpha^{IC} > 0$, the marginal rate of substitution of protection sellers is

$$\frac{u'(E[\theta|\underline{s}] + \mathcal{P} + \alpha^{IM}(p^{IM} - \mathcal{P}))}{u'\left(E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]} [\mathcal{P} + \alpha^{IM}(R - \psi - \mathcal{P})]\right)},$$

which is also strictly larger than one.

QED

Proof of Proposition 8:

First consider the case in which that $\alpha^{IM} = 0$. In that case, we have

$$\frac{u'(E[\theta|\underline{s}] + \mathcal{P} + qx^*)}{u'(E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]} \mathcal{P} - x^*)} < \frac{u'(E[\theta|\underline{s}] + \mathcal{P})}{u'(E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]} \mathcal{P})} < \frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - \psi_I(0)}$$

where the first inequality follows from $x^* > 0$ and the fact that u' is decreasing. By Proposition 5, we have that $\alpha^* = 0$. Therefore $\alpha^* = \alpha^{IM} = 0$, and correspondingly $p^* = p^{IM}$.

Second consider the case in which $\alpha^{IM} > 0$. Since equilibrium price decreases in α , it suffices to prove that $\alpha^{IM} > \alpha^*$. There are two possibilities: Either $\alpha^* = 0$, implying that $\alpha^{IM} > \alpha^*$, or $\alpha^* > 0$. In the latter case, α^* is the root of

$$\frac{u'(E[\theta|\underline{s}] + \alpha(R - [\psi_I(\alpha) + \alpha\psi'_I(\alpha)]) + (1 - \alpha)\mathcal{P} + q^*x^*)}{u'(E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]}[\alpha(R - \psi) + (1 - \alpha)\mathcal{P}] - x^*)} = \frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - (\psi_I(\alpha) + \alpha\psi'_I(\alpha))}, \quad (93)$$

while α^{IM} is the root of

$$\frac{u'(E[\theta|\underline{s}] + \alpha(R - [\psi_I(\alpha) + \alpha\psi'_I(\alpha)]) + (1 - \alpha)\mathcal{P})}{u'(E[\theta|\bar{s}] - \frac{\Pr[\underline{s}]}{\Pr[\bar{s}]}[\alpha(R - \psi) + (1 - \alpha)\mathcal{P}])} = \frac{\frac{\psi}{1-\mu} - \psi}{\frac{\psi}{1-\mu} - (\psi_I(\alpha) + \alpha\psi'_I(\alpha))}. \quad (94)$$

The two equations are very similar. They have the same right-hand side, which is an increasing function of α . The equilibrium α is such that this right-hand side intersects the left-hand side, (93) for complete markets and (94) for incomplete markets, respectively. Note further that the left-hand side of (93) is lower than the left-hand side of (94). Consequently, the intersection of the left- and right-hand sides occurs for lower α in (93) than in (94). Hence $\alpha^* < \alpha^{IM}$.

QED