CAPITAL ALLOCATION

by

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Abstract

We demonstrate that financial firms should allocate capital to lines of business based on marginal default values. The marginal default value for a line of business is the derivative of the value of the firm's option to default with respect to the scale of the line. Marginal default values give a unique allocation of capital that adds up exactly, regardless of the joint probability distribution of returns. Capital allocations follow from the conditions for the bank's optimal portfolio. The allocations are systematically different from allocations based on VaR or contribution VaR. We also show how regulation based on risk-weighted capital requirements distorts a bank's investment decisions, even if regulatory arbitrage can be eliminated.

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1. Introduction

This paper presents a general procedure for allocating capital for financial firms. The capital allocation for a line of business should depend on its *marginal default value*, which is the derivative of the value of the firm's option to default (the "default put") with respect to a change in the scale of the business. Marginal default values are unique and add up exactly. We use marginal default values to calculate efficient capital allocations.

Capital is one of the principal determinants of credit quality. The more capital a firm has, the lower the value of its option to default and the higher its credit quality, other things being equal. Credit quality clearly matters to lenders. It can matter to other liability holders, too, including customers, employees, and contract counterparties.

Consider a financial firm that diversifies across different activities and asset classes ("lines of business"), which may include lending, trading and market making, investment banking, asset management, and retail services, such as credit-card operations. Financing comes from debt (including deposits if the firm is a bank) and from risk capital, which is primarily common equity. The capital has to be sufficient to satisfy lenders and other liability holders. If the firm is subject to prudential regulation, it has to carry enough capital to satisfy regulators, too.

If the firm can identify capital requirements by lines of business, then it can allocate its overall capital back to the businesses. If insufficient capital is allocated to risky lines of business, then the risks that these businesses impose on the firm as a whole (or on lenders or third-party guarantors) will not be properly accounted for. Capital allocation is also important if capital is costly, say because of taxes, information problems, agency costs, or if capital is limited. Limited capital has a shadow price if the firm is forced to pass up positive-NPV investments.

Capital allocation is required to assess the cost and profitability of each line of business and to set incentives and compensation correctly. It is also relevant for pricing. If capital is costly, then the more capital a product or service requires, the higher the break-even price. We show how capital allocations should be "priced" and charged back to lines of business.

Capital allocation is also necessary to calculate the net benefits of hedging or securitization. For example, suppose that a credit-default swap transaction can cancel out 20% of the risk of a bank's loan portfolio, freeing up 20% of the capital that the portfolio would otherwise require. The bank must then compare the costs of hedging to the value of the capital released. To do that, the bank has to know how much capital was properly allocated to the loan portfolio in the first place.

Our capital allocation procedure works for any joint probability distribution of returns. The key assumption is complete markets, so that individual lines of business have well-defined market values. We derive capital allocations from analysis of the conditions for an optimal portfolio of the lines of business that the firm can invest in.

Capital allocations based on marginal default values differ systematically from allocations based on value at risk (VaR). For example, capital allocations proportional to VaR would allocate zero capital to a risk-free asset. Our capital-allocation procedure allocates *negative* capital to risk-free and some low-risk assets. Such assets should not be charged for capital; they should be rewarded, because at the margin they *reduce* the value of the firm's default put.

Our analysis of capital-allocation procedures also has implications for bank regulation based on risk-based capital requirements. The problems of regulatory arbitrage are widely recognized.[1] These problems follow from the temptation for banks to substitute riskier for safer assets within each risk class. We show, however, that risk-weighted capital requirements can distort bank investment decisions, even when the regulator has full information and can shut down risk-shifting within risk classes.

Suppose the regulator wants to limit the size of a bank's default put, but that the bank would rather increase it. The regulator identifies capital allocations according to our procedures. Of course the allocations depend on the risk of each line of business and the correlation of risks across businesses. Then the regulator sets line-by-line capital requirements equal to the allocations. The capital requirements are now no longer conditions for achieving the bank's optimal portfolio, but *constraints* on the optimization. Thus the solution to the bank's portfolio problem can change in ways that frustrate the regulator's intent.

As a practical matter, we conclude that bank regulation based on risk-weighted capital requirements will always distort a bank's decisions about its optimal portfolio of lines of business, if capital is costly and the constraint on credit quality or default-put value is binding.

If a regulator really had the information required to set ideal risk-based capital requirements for every one of a bank's lines of business, the regulator would not need the line-by-line requirements. It would have sufficient information to enforce a constraint on the bank's *overall* credit quality or default-put value. The bank could then solve its

[1] Palia and Porter (2003) include a good description of risk-weighted capital requirements and regulatory capital arbitrage.

optimal portfolio and capital allocations subject to the constraint, and would allocate capital efficiently. Risk-based capital requirements would be redundant.

Perhaps our results about capital allocation and regulation would be just incremental if the theory of capital allocation were well understood. But a literature search for general principles does not yield clear answers. Allocations based on value-at-risk (VaR) ignore diversification across lines of business and therefore do not add up properly. The proper procedures for allocating this diversification benefit are not obvious, and it appears that various procedures are used in practice.[2] Contribution VaRs, which depend on the covariances or betas of line-by-line returns vs. returns for the firm as a whole, do add up. See Saita (1999) and Stulz (2003), for example.[3] But we will show why allocations based on contribution VaRs are not consistent with optimizing the portfolio of lines of business.

The academic and applied literature on VaR and risk management is enormous. See Jorion (2006) and Stulz (2003), for example. Prior work on capital allocation is much more limited. Merton and Perold (1993) and Perold (2005) are probably the best places to start. These papers focus on decisions to add or subtract an entire line of business, and conclude that a bank should not attempt to allocate its total capital back to lines of business. We disagree with this conclusion, but agree with how they set up the capital-allocation problem. They define "risk capital" as the present-value cost of

[2] See Helbekkmo (2006), for example.

[3] Contribution VaRs appear in Froot and Stein (1998, pp. 67-68), Stoughton and Zechner (2007), Saita (1999), Stulz (2003, pp. 99-103) and no doubt in other places. The label varies: synonyms for "contribution" include "marginal," for example in Saita and Stulz. Others refer to "incremental VaR," which is not the same thing. Incremental VaR is the discrete change in VaR from adding or subtracting an asset or business from the bank's overall portfolio. Merton and Perold (1993), Perold (2005) and Turnbull (2000) focus on incremental VaR.

acquiring complete credit protection for the firm. We start with the present value of the bank's default put, which is the same thing.

Froot and Stein (1998) consider capital allocation, but their main interest is how banks invest capital, not how to allocate an existing stock of capital to a portfolio of existing businesses. They show that value-maximizing banks will act as if risk-averse, even in perfect financial markets, if investment opportunities are uncertain and raising equity capital on short notice is costly. They discuss contribution VAR and the problems of implementing risk-adjusted return on capital (RAROC). They do not consider default, however.[4] Turnbull (2000) extends this line of research, introducing default risk. Stoughton and Zechner (2007) add a focus on information and agency costs internal to the firm.[5]

This paper is not about the optimal level of capital, either from a private or social point of view.[6] We do assume that equity capital is costly for tax reasons or possibly because of agency costs. But we do not model the costs of default or financial distress explicitly.

This paper extends Myers and Read (2001), who analyze capital (surplus) allocation for insurance companies.[7] Principles are similar here, although the application is different and proofs are more general. Myers and Read considered capital allocation for a fixed portfolio of lines of insurance with joint lognormal or normal distributions. Here we derive capital allocations when a bank or other financial firm chooses its *optimal*

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[4] Froot (2007) builds on this model to analyze risk allocation in the insurance industry.

[5] Recent articles on capital allocation in the insurance literature include Zanjani (2010) and Bauer and Zanjani (2011).

[6] See Kashyap et al. (2010) and Miles et al. (2011).

[7] Follow-on articles in the insurance literature include Cummins, Lin, and Phillips (2006) and Grundl and Schmeiser (2007).

portfolio of lines of businesses. Our procedures work for any joint probability distribution for the relevant lines of business.

Section 2 of this paper introduces our approach and shows how marginal default values depend on capital allocations. Section 3 derives the bank's optimal portfolio of businesses as the solution to a mathematical programming problem. Capital allocations follow from the conditions for the optimum portfolio. We show how capital allocations should be priced and charged back to lines of business. We derive an adjusted present value (APV) rule for valuing investment in a line of business.

Section 4 presents examples of portfolio optimization and capital allocation assuming that asset returns are jointly normally distributed. Section 5 considers implications for bank regulation. Section 6 recaps the paper's main findings and notes areas for further work.

2. Default Values and Capital Allocation

Our results apply not only to banks but to financial firms generally. A firm's market-value balance sheet is:

<u>Assets</u>	<u>Liabilities</u>
Assets ($A = A_1 \ldots A_M$)	Debt (D)
Default Put (P)	Equity (E)
Franchise value (G)	

The lines of business A_1 to A_M are assumed to be marked to market. The firm's "franchise value," which includes intangible assets and the present value of future growth opportunities, is entered as G. We assume that G disappears if the firm defaults in the current period. That is, G = 0 in bankruptcy. We doubt that any material fraction of Lehman Brothers' franchise value survived its bankruptcy, for example.

The *default-risk free* value of debt, including deposits, is D. Default risk is captured not in the value of debt, but on the other side of the balance sheet as the default value P, the present value of the bank's default put over the next period. For purposes of capital allocation, a period could be a month, quarter or year, but probably not longer.

If the firm defaults, the value of the default put equals the shortfall of the end-of-period asset value from the end-of-period debt value, including interest due. Lenders, customers and other liability holders do not necessarily bear this shortfall. The put payoff may be covered, at least in part, by deposit guarantees or other third-party credit backup. Who bears losses in default does not necessarily matter for our analysis, however. We do not need to model deposit insurance explicitly if we assume that any costs of the insurance or other forms of credit backup are sunk and already paid for.

Equity (E) is the market value of the bank's equity, common stock plus issues of preferred stock or subordinated debt that count as capital. The bank's capital C is not the same thing as its equity, however. The capital-account balance sheet is:

Assets	Liabilities
Assets (A)	Debt (D)
	Capital (C)

Capital is $C = A - D$, the difference between the market value of the firm's assets and the *default-risk free* value of its debt and deposits.[8] In practice, some of the firm's

[8] Casual misunderstandings of "capital" are common. Capital is not a pot of cash held in a reserve account or money market securities. It is a measure of the investment in equity (or in some types of subordinate debt) that protects debt and deposits. Capital is not entirely marked to market. If it were, a firm could

assets may not be marked to market. The important distinction here, however, is that capital does not incorporate default value P or the intangible assets or future growth opportunities in G.

The asset portfolio consists of M assets (lines of business) with start-of-period values A_i. Thus $A = \sum_{i=1}^{M} A_i$. The value of the default put is:

$$P = PV[\max\{0, (R_D D - R_A A)\}] \quad (1)$$

where R_D is the gross return to a dollar of debt or deposits (one plus a safe interest rate) and R_A is the uncertain gross return on the bank's assets. All returns are assumed to be uncertain except for R_D.[9] The end-of-period promised payoff to debt and deposits, including interest, is $R_D D$. With complete markets, the present value of the default put is:

$$P = \int_Z [R_D D - R_A A]\pi(z)dz \quad (2)$$

where $\pi(z)$ is a state-price density in the default region Z. This region consists of all outcomes where assets fall short of liabilities and hence the put is "in the money."

Each state z is a unique point in the default region Z. Each point is a combination of returns on the assets (R_i), which generate a portfolio return of $R_A A$. The valuation Eq. (2) sums across the continuum of states, with the payoff in each state z multiplied by the state-price density $\pi(z)$. Note that the states are identified by asset returns and that the state prices $\pi(z)$ are fixed. Therefore an extra dollar delivered in state z by asset A_i has

increase its capital simply by increasing asset risk and the risk of default, thus decreasing the market value of the firm's debt. Our definition of capital does *not* mark risky debt and deposits to market.

[9] We take D, the face amount of debt and deposits, as fixed. $R_D D$ is the exercise price of the default put. We could allow for uncertain liabilities, for example insurance contracts, as in Myers and Read (2001). But in this context it's easier to think of a risky liability as a short position in a risky asset.

exactly the same present value as an extra dollar delivered by A_j. The valuation formula sums across states.

Define the *marginal default value* of asset *i* as $p_i = \partial P / \partial A_i$, the partial derivative of overall put value P with respect to A_i.[10] We can show that these marginal default-option values add up uniquely. The sum of the products of each asset and its marginal default value equals the default value of the bank as a whole:

Proposition 1. The default value P can be expressed as an asset-weighted sum of marginal default values p_i:

$$P = \sum_{i=1}^{M} p_i A_i \tag{3}$$

A proof for the two-asset case is provided in Appendix 1. Generalization to M assets is straightforward. Note that Eq. (3) requires no assumptions about the probability distribution of returns. The only assumption is complete markets, so that the assets A_i have well-defined market values.

The capital ratio for the bank as a whole is $c \equiv \frac{C}{A}$. Therefore, $D = (1 - c)\,A$ and Eq. (2) can be modified as:

$$P = \int_Z A[R_D(1 - c) - R_A]\pi(z)dz \tag{4}$$

It's clear from this valuation formula that an across-the-board expansion of assets and liabilities (with *c* constant) will result in a proportional increase in overall default

[10] Note that Myers-Read (2001) define marginal default values with respect to liabilities, whereas this paper defines marginal default values with respect to assets.

value. Given c, $p \equiv \dfrac{\partial P}{\partial A}$ is a constant for any proportional change, regardless of the size of the change.

Expansion of a single line of business will also affect P, but not proportionally. Therefore we also allow capital ratios to vary by line. Define the capital ratio for line i as c_i. Default value is:

$$\begin{aligned} P &= \int_Z \left[\left(\sum_i (1-c_i) A_i \right) R_D - \left(\sum_i A_i R_i \right) \right] \pi(z) dz \\ &= \sum_i \int_Z A_i \left[(1-c_i) R_D - R_i \right] \pi(z) dz \end{aligned} \tag{5}$$

The default value per unit of assets is

$$p = \frac{P}{A} = \sum_i \int_Z a_i \left[(1-c_i) R_D - R_i \right] \pi(z) dz, \tag{6}$$

where $a_i \equiv \dfrac{A_i}{A}$. The marginal default values are

$$p_i \equiv \frac{\partial P}{\partial A_i} = \int_Z \left[(1-c_i) R_D - R_i \right] \pi(z) dz = \frac{\partial p}{\partial a_i} \tag{7}$$

Our adding-up result still holds. Also, an increase in the marginal capital allocation c_i always decreases the exercise price of bank's default put and reduces its value. Therefore, we can offset differences in p_i by compensating changes in the capital allocations c_i.

We derive optimal capital allocations in the next section. But Eq. (7) gives a preview of one result. For a risk-free asset, where $R_i = R_D$, marginal default value is negative at any positive capital allocation c_i.

$$\mathrm{p_i} = \int_{\mathrm{Z}} [-\mathrm{c_i R_D}]\pi(\mathrm{z})\mathrm{dz} < 0. \tag{8}$$

We will show in the next section that optimal capital-adjusted marginal default values must be all positive and equal across lines. Thus risk-free assets must be given a *negative* capital allocation c_i. Other low-risk assets may also get negative allocations. Note the contrast to allocations based on VaR or contribution VaR. For example, the contribution VaR for a safe asset is zero, since the covariance of the safe return with the bank's overall return is zero.

If capital allocations are constant ($c_i = c$), marginal default values p_i will vary across lines of business. A bank that allocates capital in proportion to assets, despite varying marginal default values, is forcing some businesses to cross-subsidize others. This contaminates performance measurement, incentives, compensation, pricing and decisions about securitization and hedging. The remedy is to vary capital allocation depending on marginal default values, so that each business's capital-adjusted contribution to default value is the same.

Before moving in the next section to the bank's optimal portfolio problem, we note two further results. First, marginal default values can be expressed as the sum of a scale term and a business-composition term:

$$p_i \;=\; p + \frac{\partial p}{\partial a_i}\left(1 - a_i\right) \tag{9}$$

where $a_i \equiv \dfrac{A_i}{A}$. The first term p is the change in default value due to an increase in A, the overall scale of the bank's assets, ignoring any change in the composition of its assets. The second term captures the change in p due to a change in the composition of the asset portfolio $\partial p/\partial a_i$. The partial derivatives of the unit default value p and the marginal default values p_i with respect to the allocations c_i are:

$$\frac{\partial p}{\partial c_i} = -\int_Z a_i R_D \pi(z)\,dz = a_i\left(\frac{\partial p}{\partial c}\right) \tag{10}$$

$$\frac{\partial p_i}{\partial c_i} = -\int_Z R_D \pi(z)\,dz = \frac{\partial p}{\partial c} \tag{11}$$

Second, the valuation expressions can be simplified by defining $\Pi_z(R_x) \equiv \int_Z R_x \pi(z)dz.$. For example, $\Pi_z(R_D)$ is the present value of a safe asset's return (but only in the in-the-money region Z, like the payoff on a cash-or-nothing put triggered by default). Write marginal default value p_i as:

$$p_i = (1 - c_i)\Pi_Z(R_D) - \Pi_Z(R_i). \tag{13}$$

The overall default value is

$$p = (1 - c)\Pi_Z(R_D) - \Pi_Z(R_A). \tag{14}$$

Here $(1 - c)\Pi_z(R_D)$ is the present value of the exercise price of the default put, received only if the put is exercised. $\Pi_Z(R_A)$ is the present value of the asset given up if the put is exercised. The difference between these two values is the value of the put.

The present values $\Pi_Z(R_X)$ have exact analytic solutions if returns are normally distributed. Our results and procedures do not depend on specific probability distributions, however, so we use this more general notation.

Next we consider how a financial firm should (in principle) choose its optimal portfolio of lines of business. Capital allocation formulas will follow from the conditions for the optimum.

3. Portfolio Optimization and Capital Allocation

We consider a setting where management searches for the optimum *portfolio* of candidate lines of business. This is a mathematical programming problem. Banks and other financial firms solve this problem implicitly when they set strategy or launch takeovers or major restructurings.

The programming problem has two constraints: a constraint on capital and a constraint the size of the default put. The capital constraint is straightforward, but the put constraint can be specified in at least two ways. In section 3.1, we define the put constraint in terms of *credit quality*, specifically as the maximum permissible *ratio* of default value to the value of default-free liabilities. A constraint on this ratio is equivalent to a constraint on the maximum allowable credit spread in the interest rate demanded by lenders or depositors who cannot count on the government or some other third party to bail them out in the event of default. This constraint is needed to identify capital allocations that avoid cross subsidies among lines of business. In section 3.2, we constrain the maximum *dollar value* of the default put. This value is what regulators or credit insurers should care about if they may have to bail out lenders and depositors. Of course these constraints could also be adopted voluntarily by a bank anxious to preserve its franchise value and growth opportunities. They could also be imposed together.

We assume throughout that the objective is to maximize the market value of the firm, subject to constraints on capital and default value. Capital has a tax cost of τ per dollar per period, because returns to equity investors are not tax-deductible. (For simplicity we will just refer to tax costs of equity, but τ could also cover other costs of contributing and maintaining capital, including agency costs and costs of monitoring by outside investors.) The adjusted present value of business *i* (ignoring the constraints) is $APV_i = NPV_i - \tau c_i A_i$.[11] For simplicity we assume the optimal portfolio is chosen once or for one period only.[12]

3.1. Portfolio Optimization with a Credit Quality Constraint

We assume that debt can be raised at market rates in zero-NPV transactions. That is, we assume that debt markets are competitive and that lenders and depositors are fully informed. Thus the firm must pay interest rates that fairly compensate lenders for the default risk they bear.[13] [14] In other words, the firm pays for 100% of the default put value. It provides credit assurance through its balance sheet, not through financial guarantees supplied by a third party.

Define the credit-quality constraint as the ratio of the present value of the default put to the default-risk-free value of debt (including deposits). The constraint is not on the

[11] In corporate finance, the tax-adjustment term in APV is usually expressed as the tax advantage of debt vs. equity. NPV is calculated at an opportunity cost of capital, as if the investment were all-equity financed, and the present value of interest tax shields is then added. See Myers (1974) and Brealey, Myers and Allen (2010), Ch. 19. The interest tax shields depend on the amount of debt supported by the investment. In our setting, where the firm is allocating capital, NPV should be calculated as if the investment were 100% financed by deposits and other debt. The tax cost of the capital required to support the investment should then be subtracted. These alternatives are of course equivalent, two sides of the same coin.

[12] We hold franchise value and growth opportunities G constant. Dynamics are more complicated. Froot and Stein (1998) introduce some dynamics of bank capital structure decisions.

[13] The interest rate paid to depositors includes the value of transaction and other services provided “free of charge” by the bank.

total default value, but on the default value *per unit* of liabilities. Assume that management must keep this default ratio at or below α, that is, $P \leq \alpha D$, which is equivalent to $p \leq \alpha(1-c)$.

The decision variables are A_i, the amounts of assets $i = 1 \ldots M$ for the firm's M lines of business, plus the amount of capital C and the amount of debt D raised from lenders. Assets can be acquired for a fraction $e_i < 1$ of their current market values, so $NPV_i = (1-e_i)A_i$.[15] [16] Thus we assume all lines have positive NPVs and, for simplicity, constant returns to investment. We also assume that the combined NPVs exceed the cost of capital, τC, so that the capital and credit-quality constraints are binding at the optimum. Financing must cover investment, so $\sum(1-e_i)A_i < C+D$. We omit this constraint, however, because borrowing is zero-NPV and the constraint would have a shadow price of zero.

The default put value P depends on the scale and mix of assets, the joint probability distribution of asset returns, and on the amount of capital C. The constraint $P \leq \alpha D$ therefore sets the floor for C. That is, for any given mix of business, there is a minimum capital ratio consistent with the credit quality constraint α. This constraint will be binding at the optimum, with a shadow price $\lambda_\alpha > 0$. Capital is fixed at $\overline{C}$, so $\sum c_i A_i \leq \overline{C}$, where c_i is the capital allocation per dollar invested in asset A_i. This constraint will also be binding at the optimum ($\lambda_c > 0$).

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[14] We could also assume here that the firm has deposit insurance but must pay a premium sufficient to make the insurance zero-NPV.

[15] For simplicity we assume constant returns to scale. Scale of investment is limited by the increase in marginal default value when a line of business grows to a greater fraction of the overall portfolio.

The capital ratio for the firm is a weighted average of the line-of-business capital ratios: $c = \sum_i c_i a_i$, where $a_i \equiv \frac{A_i}{A}$. We showed earlier that $P = \sum_i p_i A_i$. Then, with constraints on credit quality ($p \le \alpha(1-c)$) and capital ($C \le \overline{C}$), we can write the mathematical programming problem as follows. The Lagrangian is:

$$L(A_i, c_i, \lambda_\alpha, \lambda_c) = \sum_i A_i(1 - e_i - \tau c_i) + \lambda_\alpha\left(\alpha \sum_i (1-c_i)A_i - \sum_i p_i A_i\right) + \lambda_c\left(\overline{C} - \sum_i c_i A_i\right) \quad (15)$$

The conditions for an optimum are

(15a) $$\frac{\partial L}{\partial A_i} = 1 - e_i - (\tau + \lambda_c)c_i + \lambda_\alpha(\alpha(1-c_i) - p_i) = 0$$

(15b) $$\frac{\partial L}{\partial c_i} = -A_i\left[(\tau + \lambda_c) + \lambda_\alpha\left(\alpha + \frac{\partial p_i}{\partial c_i}\right)\right] = 0$$

(15c) $$\frac{\partial L}{\partial \lambda_\alpha} = \alpha(1-c)A - P = 0$$

(15d) $$\frac{\partial L}{\partial \lambda_c} = \overline{C} - C = 0$$

Condition (15a) can also be written as

$$\frac{\partial L}{\partial A_i} = 1 - e_i - c_i\left[\tau + \lambda_C + \lambda_\alpha\left(\frac{\alpha(1-c_i) - p_i}{c_i}\right)\right] = 0 \quad (16)$$

Eq. (16) implies that the firm's all-in cost per dollar of capital is $\tau + \lambda_c + \lambda_\alpha\left(\frac{\alpha(1-c_i) - p_i}{c_i}\right)$. This all-in cost of capital applies to the firm as a whole,

[16] Asset returns can depend on credit quality, for example because of collateral requirements or costs imposed by nervous counterparties. Thus e_i could depend on α. But credit quality is held constant in this optimization so long as the credit quality constraint is binding.

not to individual lines of business.[17] At an optimum it will be the same for all lines of business. Therefore, capital allocations should be set line by line so that they vary with marginal default values as follows:[18]

$$\frac{p_1}{1-c_1} = \frac{p_2}{1-c_2} = \cdots = \frac{p}{1-c} = \alpha \qquad (17)$$

Recall from Eq. (11) that $\frac{\partial p_i}{\partial c_i} = \frac{\partial p}{\partial c}$, so condition (15b) for the optimum can be written as

$$\frac{\partial L}{\partial c_i} = -A_i\left[\left(\tau+\lambda_c\right) + \lambda_\alpha\left(\alpha+\frac{\partial p}{\partial c}\right)\right] = 0 \qquad (18)$$

This result says that if capital allocations are set correctly, it will not be possible to add value simply by reallocating capital from one line of business to another. Note that at an optimum the all-in cost of capital is $\tau+\lambda_C$ because $\alpha\left(1-c_i\right)-p_i=0$. Note too that the firm might repeat this optimization exercise for several different values of α, for example, once for AAA credit quality, once for AA quality, etc., in order to identify the most profitable credit quality at which to operate its business.

Eq. (17) is not yet a recipe for calculating capital allocations, because the marginal default value p_i depends in part on c_i, the capital allocation to line i, which we have not yet determined. Eq. (17) just says that capital-adjusted marginal default values must all be the same when expressed as a fraction of liabilities. (Dividing p_i by $1-c_i$

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[17] Note that the cost of capital is a marginal dollar amount, not a percentage to be added to an interest rate.

[18] This result may appear inconsistent with Myers-Read (2001), who conclude that marginal default values in all lines of business must be equal to avoid cross subsidies. But they define marginal default values with respect to liabilities, not assets. Eq. (17) is equivalent to the condition that marginal default values with respect to liabilities are the same in all lines.

gives the ratio of marginal default value to the debt used at the margin to finance assets in line of business *i*.)

3.1.1. Capital Allocation with the Credit Quality Constraint

Recall from Eqs. (13) and (14) that $p_i = (1-c_i)\,\Pi_Z(R_D) - \Pi_Z(R_i)$ and $p = (1-c)\,\Pi_Z(R_D) - \Pi_Z(R_A)$. Therefore, the relationship between the marginal default value for a line of business and the default value for the firm as a whole is

$$p_i - p = -(c_i - c)\,\Pi_Z(R_D) - \left(\Pi_Z(R_i) - \Pi_Z(R_A)\right) \tag{19}$$

Also, we know from Eq. (16) that capital allocations should be set line by line so that $\frac{p_i}{1-c_i} = \frac{p}{1-c} = \alpha$. Putting these results together gives the following formula for allocating capital by line of business:

$$c_i = c + \frac{\Pi_Z(R_A) - \Pi_Z(R_i)}{\Pi_Z(R_D) - \alpha} \tag{20}$$

Marginal capital allocations for an asset or line of business depend on the present value of its returns in default, that is, on the present value of its returns as distributed across the default region Z. If its returns are "riskier" than the overall portfolio return R_A in region Z—that is, worth less than the overall portfolio return in that region—then $c_i > c$. If its returns are relatively “safe” in region Z—worth more than the overall return in default—then $c_i < c$. The capital ratio for line i does *not* depend on the line's marginal effect on the *probability* of default. It depends on the *value* of the line's payoff in default.

Thus capital can be allocated depending on the marginal default value of each line of business, where marginal default value is the derivative of the value of the firm's default put with respect to a change in the scale of the business. Marginal default values give a unique allocation that adds up exactly. Differences in marginal default values can be offset by differences in marginal capital allocations. Cross subsidies are avoided if capital allocations are set so that capital-adjusted marginal default values are the same for all lines, as in Eq. (17). Each line's capital ratio should depend on the value of the line's payoffs in default. The procedure of setting marginal capital requirements to equalize capital-adjusted marginal default values follows from optimization of the firm's portfolio of businesses.

3.2. Portfolio Optimization with a Constraint on Default Value

Now we reconsider the firm's optimal portfolio when the *total* default value is constrained. This could be the relevant constraint for a bank regulator or deposit insurer that must cover losses if the bank defaults. Such a guarantor would limit its total exposure in default. (If instead it limited the default put per dollar of liabilities, the firm could expand its insurance coverage just by expanding and borrowing more.)[19]

The default put value P in this case depends as before on the scale and mix of assets, the joint probability distribution of the assets' returns, and on the amount of capital C. The constraint $P \le \overline{P}$ now sets the floor for C. The constraints on C and P will be binding at the optimum ($\lambda_p > 0$ and $\lambda_c > 0$) if the candidate lines of business are sufficiently profitable. The Lagrangian is

$$L(A, c, \lambda_p, \lambda_c) = \sum_i A_i (1 - e_i - \tau c_i) + \lambda_P \left(\overline{P} - \sum_i p_i A_i \right) + \lambda_C \left(\overline{C} - \sum_i c_i A_i \right) \qquad (22)$$

The conditions for an optimum are

(22a) $$\frac{\partial L}{\partial A_i} = 1 - e_i - (\tau + \lambda_C) c_i - \lambda_P p_i = 0$$

(22b) $$\frac{\partial L}{\partial c_i} = -A_i \left[(\tau + \lambda_C) + \lambda_P \left(\frac{\partial p_i}{\partial c_i} \right) \right] = 0$$

(22c) $$\frac{\partial L}{\partial \lambda_P} = \overline{P} - P = 0$$

(22d) $$\frac{\partial L}{\partial \lambda_C} = \overline{C} - C = 0$$

Condition (22a) can be rewritten as follows:

$$\frac{\partial L}{\partial A_i} = 1 - e_i - c_i \left[\tau + \lambda_C + \lambda_P \left(\frac{p_i}{c_i} \right) \right] = 0 \qquad (23)$$

Eq. (23) implies that the all-in cost of capital is $\tau + \lambda_C + \lambda_P \left(\frac{p_i}{c_i} \right)$. At an optimum it will be the same in all lines of business. Therefore capital allocations should be set so that $\frac{p_i}{c_i} = \frac{p}{c}$ for each line of business *i*. Note that in this case, with the constraint on the total

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[19] It would perhaps be more sensible to constrain *both* credit quality *and* the value of the default put—or to solve for the amount of capital necessary to keep the value of the default put at or below a maximum while

default value rather than the default value per unit of liabilities, the third term in the all-in cost of capital does not vanish at the optimum.

From Eq. (11), we know that $\frac{\partial p_i}{\partial c_i} = \frac{\partial p}{\partial c}$ for each asset.[20] Thus condition (22a) can be written as:

$$\frac{\partial L}{\partial c_i} = -A_i\left[(\tau + \lambda_C) + \lambda_P\left(\frac{\partial p}{\partial c}\right)\right] = 0 \tag{24}$$

As in the preceding case, this condition says that it will not be possible to increase profitability simply by reallocating capital.

3.2.1. Capital Allocation with a Constraint on Default Value

Recall Eq. (19), the relationship between the marginal default value for a line of business and the marginal default value for the firm as a whole:

$$p_i - p = -(c_i - c)\,\Pi_Z(R_D) - \left(\Pi_Z(R_i) - \Pi_Z(R_A)\right)$$

We concluded in this case that capital allocations should be set so that $\frac{p_i}{c_i} = \frac{p}{c}$. This implies that $p_i - p = \left(\frac{p}{c}\right)(c_i - c)$. When we put these results together, we get the following formula for allocating capital:

imposing a minimum credit quality constraint.

[20] Note that condition (22b) implies that $\frac{\partial p_i}{\partial c_i} = -\frac{(\tau + \lambda_C)}{\lambda_P}$ is a constant, so that each line's capital-adjusted contribution to the default value should be the same. In other words, the firm should adjust capital allocations so that the contribution of each asset to the total put value is the same constant. Otherwise the firm's portfolio could not be optimal; portfolio value could be increased by investing less in assets with

$$c_i \;=\; c \;+\; \frac{\Pi_Z(R_A) - \Pi_Z(R_i)}{\Pi_Z(R_D) \;+\; \dfrac{p}{c}} \tag{25}$$

The capital-allocation formulas in Eqs. (20) and (25) differ only in the denominator of the right-hand terms. With a credit quality constraint, the denominator is $\Pi_Z(R_D)$ - α, where α is P/D, the ratio of put value to debt. With a constraint on the dollar value of the put, α is replaced by P/C, the ratio of put value to capital.

Thus we have derived two capital allocation rules from the firm's optimization of its portfolio of businesses, one rule incorporating a constraint on credit quality and another rule incorporating a constraint on the total default value. These rules can be used to allocate the tax or other costs of capital back to lines of business subject to the respective constraints. Examples follow in the next section.

Before proceeding, we note that (absent regulation) a financial firm would almost surely choose its optimal portfolio and allocate capital subject to a credit quality constraint. Credit quality is relevant to the valuation and pricing of all liabilities, not just debt or deposits. Many financial firms have credit-sensitive lines of business, including insurance companies and swap dealers. Such firms face a feedback from credit quality to profitability. For example, lower credit quality may trigger higher collateral requirements, as AIG found out. An "optimal" portfolio subject to a constraint on the dollar value of the default put misses this feedback loop.

4. Default Values and Capital Allocation for Joint Normal Distributions

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high dp_i/dc_i and more in assets with low dp_i/dc_i. This observation also leads us to the result in Eq. (11) that $\frac{\partial p_i}{\partial c_i} = \frac{\partial p}{\partial c}$ for each asset.

If asset returns are normally distributed, the return to a portfolio of assets is normally distributed too. This allows closed-form formulas for marginal default values and capital allocations. The formulas illustrate how our capital allocation procedures work and what the allocations depend on. The formulas will also make it relatively easy to construct and interpret numerical examples.

The default value P depends on the market value of assets A, the market value of default-free liabilities D, and on σ_A, the standard deviation of end-of-period asset returns per unit of assets. The present value of the default option is:

$$P(A, D, \sigma_A) \;=\; (D - A)\,\mathrm{N}\{y\} + \sigma_A A\,\mathrm{N}'\{y\} \tag{26}$$

where N is the cumulative distribution function for a standard normal variable and $y = \dfrac{D - A}{\sigma_A A}$. Capital is defined as the market value of assets less the market value of default-free liabilities, so the present value of the default option can also be expressed as a function of assets and capital:

$$P(A, C, \sigma_A) \;=\; -C\,\mathrm{N}\{y\} + \sigma_A A\,\mathrm{N}'\{y\} \tag{26a}$$

where $y \;=\; \dfrac{C}{\sigma_A A}$. Finally, the default value per unit of assets can be expressed as a function of the capital-to-asset ratio:

$$p\,(c, \sigma_A) \;=\; -c\,\mathrm{N}\{y\} + \sigma_A\,\mathrm{N}'\{y\} \tag{26b}$$

where $y \;=\; \dfrac{c}{\sigma_A}$.

Remember from Eq. (9) that

$$p_i \;=\; \frac{\partial P}{\partial A_i} \;=\; p + \frac{\partial p}{\partial a_i}(1 - a_i)$$

The change in p due to a change in the composition of the portfolio ($\partial p/\partial a_i$) can also be written as $\frac{\partial p}{\partial a_i} = \frac{\partial p}{\partial c}\frac{\partial c}{\partial a_i} + \frac{\partial p}{\partial \sigma_A}\frac{\partial \sigma_A}{\partial a_i}$. But $\frac{\partial c}{\partial a_i} = \frac{c_i - c}{1-a_i}$ and $\frac{\partial \sigma_A}{\partial a_i} = \frac{\sigma_{iA} - \sigma_A^2}{\sigma_A(1-a_i)}$, so the marginal default value for each line of business is:

$$p_i = p + \frac{\partial p}{\partial c}(c_i - c) + \frac{\partial p}{\partial \sigma_A}\left(\frac{\sigma_{iA} - \sigma_A^2}{\sigma_A}\right) \tag{27}$$

where σ_{iA} is the covariance of the return on line of business *i* with the portfolio return. The option delta ($\partial p/\partial c$) and vega ($\partial p/\partial \sigma_A$), respectively, are

$$\frac{\partial p}{\partial c} = -\,\mathrm{N}\{y\} \tag{28a}$$

$$\frac{\partial p}{\partial \sigma_A} = \mathrm{N}'\{y\} \tag{28b}$$

The option delta is negative, so the higher the capital allocation, the lower the marginal default value. The option vega is positive, so the higher the covariance of returns, the higher the marginal default value.

When asset returns are normally distributed and we impose the constraint on credit quality, $\frac{p_i}{1-c_i} = \frac{p}{1-c} = \alpha$, we get the following capital allocation rule using Eq. (27):

$$c_i = c - \left(\frac{\partial p}{\partial c} + \alpha\right)^{-1}\left(\frac{\partial p}{\partial \sigma_A}\right)\left[\frac{(\sigma_{iA} - \sigma_A^2)}{\sigma_A}\right] \tag{29}$$

If we impose the constraint on the total default value rather than the default value per unit of liabilities, $\frac{\partial p_i}{\partial c_i} = \frac{\partial p}{\partial c}$ thus $p_i - p = \frac{\partial p}{\partial c}(c_i - c)$, we get a different capital allocation rule:

$$c_i = c - \left(\frac{\partial p}{\partial c} - \frac{p}{c}\right)^{-1}\left(\frac{\partial p}{\partial \sigma_A}\right)\left[\frac{(\sigma_{iA} - \sigma_A^2)}{\sigma_A}\right] \tag{30}$$

Note that $\frac{\partial p}{\partial c} < 0$ and $\frac{p}{c} > 0$, so the term $\left(\frac{\partial p}{\partial c} - \frac{p}{c}\right) < 0$.

Thus the marginal capital allocations in the normal case depend on σ_{iA}, the covariance of asset i's return with the overall return, relative to the variance of the overall return σ_A^2. Riskier assets ($\sigma_{iA} > \sigma_A^2$) must be allocated extra capital ($c_i > c$). Safer assets ($\sigma_{iA} < \sigma_A^2$) require less capital ($c_i < c$). Safe or low-risk assets will have negative capital allocations.

Eqs. (29) and (30) say that marginal capital allocations depend on the covariance of returns to the assets in a line of business with the returns to the firm's portfolio of asset, the standard deviation of returns to the firm's portfolio of asset, and also on the delta and vega of the default put. The optimal marginal allocations are not proportional to that covariance, as in a contribution VaR analysis, but to the *difference* between the covariance and the portfolio variance.

4.1. Numerical Examples

4.1.1. Numerical Examples with Constraint on Credit Quality

Consider a firm selecting a portfolio of businesses from scratch. The firm has a fixed amount of capital to allocate. Suppose there are only two businesses to choose from—the firm could put all of its capital into one line of business or some into both lines of business. Let $x_1 = x$ and $x_2 = 1 - x$ be the proportions of total assets invested in the two lines. Assume that asset returns are normally distributed and the firm's liabilities are riskless (aside from the possibility of default). Under these assumptions the risk of the firm's assets, σ_A, is completely determined by the standard deviations of asset returns, σ_1 and σ_2, the correlation of returns, ρ, and the asset allocation, x. For any pair of assets, the risk of the portfolio is a function of the asset allocation: $\sigma_A = \sigma_A(x)$.

Suppose first that the firm operates subject to a credit quality constraint: its default value must not exceed a fraction α of the present value of liabilities ($P \leq \alpha D$). Assume that the capital and credit quality constraints are binding ($C = \overline{C}$ and $P = \alpha D$). The minimum capital ratio c^* consistent with the credit quality constraint and the risk of the portfolio can be determined by solving Eq. (26b). Total assets A, default-free liabilities D, and default value P can be calculated based on total capital C and the capital ratio $c^*(x)$:

$$A(x) = \frac{\overline{C}}{c^*(x)}$$

$$D(x) = (1 - c^*(x))A(x)$$

$$P(x) = \alpha\, D(x)$$

We can use Eq. (27) to calculate marginal default values and Eq. (29) to determine the capital allocations.

The adjusted present value of the firm's portfolio of investments depends on the amount of assets acquired A, the net present value per dollar of investment in each line of business, the asset allocation (x), and the tax cost of capital (τ).[21] In our setup, the per-dollar net present values ("gross margins") are $\mathrm{NPV}_1 = 1 - e_1$ and $\mathrm{NPV}_2 = 1 - e_2$.[22] The APV of the firm's portfolio of investments is $\mathrm{APV}(x) = A[x(1-e_1) + (1-x)(1-e_2) - \tau c^*(x)]$. The APVs of each line of business are $\mathrm{APV}_1(x) = Ax[1 - e_1 - \tau c_1^*(x)]$ and $\mathrm{APV}_2(x) = Ax[1 - e_2 - \tau c_2^*(x)]$. The shadow price of capital λ_c (and thus the all-in cost of capital $\tau + \lambda_c$) depends on the APV, so it too depends on the asset allocation.

Let x^* be the asset allocation at which the APV of the portfolio is a maximum. At the optimum, the ratio of NPV to capital is the same in both lines of business and is equal to the cost of capital: $\dfrac{1 - e_1}{c_1^*(x^*)} = \dfrac{1 - e_2}{c_2^*(x^*)} = \tau + \lambda_c$. Also, the marginal profit in both lines—the profit including the shadow price of capital—is zero: $1 - e_1 - (\tau + \lambda_c)c_1^*(x^*) = 1 - e_2 - (\tau + \lambda_c)c_2^*(x^*) = 0$.

Assume the firm has \$1,000 of capital and maintains credit quality equal to α = 1%. The standard deviations of asset returns for Lines 1 and 2 are 10% and 20%, respectively; the per-dollar net present values are 3% and 5%, respectively; and the returns are uncorrelated. Assume the tax cost of capital is τ = 3% per period.

The numerical results are summarized in Table 1. The first column contains the results at the optimal asset allocation, where the APV of the firm's investments is at the maximum. The columns to the right give results for allocations ranging from 100% in Line 1 (0% in Line 2) to 100% in Line 2 (0% in Line 1). In all cases the constraints on

[21] Note again that τ could also cover other costs, including agency costs and cost of monitoring by investors.

capital and credit quality are binding. If the firm chooses to operate as a stand-alone Line 1 business (with asset risk equal to 10%), its capital ratio will be 9.55%. If the firm chooses to operate as a stand-alone Line 2 business (with asset risk equal to 20%), its capital ratio will be 28.16%.

The optimum asset allocation calls for investing 74.56% of assets in Line 1, which is relatively safe, and 25.44% in Line 2. Asset portfolio risk is 9.03%. The firm's capital ratio is 8.04%. Total assets are $12,439. Capital allocations are 6.87% of assets in Line 1 and 11.46% of assets in Line 2. This means $637 of capital is allocated to Line 1 and $363 of capital is allocated to Line 2. The APV of the portfolio is $259 + 147 = $406. (The APV for each line equals the NPV of investment in the line minus the cost τ times the capital allocation.)

The market value of the firm is lower at all other asset allocations. For example, with 100% of assets invested in Line 1, the APV is only $284. At this point the covariance of the return on Line 1 with the return on the portfolio is equal to the variance of the return on the portfolio—Line 1 *is* the portfolio. The capital allocation to Line 1 is just the portfolio capital ratio: 9.55%. In contrast, the covariance of the return on Line 2 with the return on the portfolio is zero. The capital allocation to Line 2 is less than zero: –6.28%. Thus, not only is the gross profitability (per-dollar net present value) of an investment in Line 2 greater than the gross profitability of an investment Line 1, the risk of marginal investment in Line 2 is less than the risk of marginal investment in Line 1.

Risk and profitability change as investment in Line 2 expands and investment in Line 1 contracts. With 90% of assets invested in Line 1 (10% in Line 2), the covariance

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[22] Again we assume constant returns to scale for simplicity. We have constructed more elaborate examples with decreasing returns to scale. Qualitative results and conclusions are unchanged.

of the return on Line 2 is greater than zero, and the covariance of the return on Line 1 is less than the variance of the portfolio return. As a result, the capital allocation to Line 2 increases from –6.28% to 0.85% and the capital allocation to Line 1 decreases from 9.55% to 9.16%. The marginal profitability of Line 2 has declined due to the increase in its marginal portfolio risk and capital allocation. (Marginal profitability for each line equals the APV of investment in the line minus the shadow price λ_c times the capital allocation. In other words, it is the marginal NPV of the line minus the all-in cost of capital, which in Table 1 is 44%.)

At the optimal asset allocation, the higher gross profitability of Line 2 is exactly offset by its higher capital cost. With capital fixed, shifting assets into Line 2 to obtain the higher gross profitability per unit comes at the cost of a reduction in total assets due to the higher capital allocation—the capital allocation necessary to satisfy the credit quality constraint. Thus the marginal profitability of Line 1 and the marginal profitability of Line 2 are zero at the optimum.

How could headquarters implement the optimal portfolio? It could simply allocate \$637 of capital to Line 1 and \$363 of capital to Line 2. Each line would be charged the 44% all-in cost of capital on any additional capital sought by either line. Neither line would want to expand, because expansion would push marginal profitability into negative territory.

If the firm started with a non-optimal asset mix, headquarters could simply charge the all-in cost of capital shown in the appropriate non-optimal column of Table 1. Notice that these costs of capital are all less than 44%. Nevertheless, one line or the other would face negative marginal profitability and move to relinquish capital, which would free up the other line to expand. Both lines would be content only at the optimum. Thus the optimum could, in principle, be achieved in a decentralized setting.

Pricing for each line's product or service would be determined by operating costs plus the all-in cost of capital at the optimum. (The 44% all-in cost is high because the example makes both lines very profitable and capital is limited to $1,000.) Compensation would be determined by the APVs, which measure the lines' contributions to firm value.

4.1.2. Numerical Examples with Constraint on Default Value

Suppose now that the firm operates subject to a constraint on its total default value rather than its default value per unit of liabilities. Now the firm chooses its portfolio subject to the constraint that the default value not exceed an amount $\overline{P}$. Assume that the capital and credit quality constraints are binding ($C = \overline{C}$ and $P = \overline{P}$). In this case, because the risk of the portfolio is completely determined by the asset allocation decision, Eq. (26a) can be solved to find the maximum amount of assets consistent with the capital and default value constraints.

Assume again that capital is fixed at $1,000 and that the maximum default value is $95. We pick $95 as the default value constraint because $95 is the default value of the firm when 100% of assets are allocated in Line 1 and the default value per unit of assets is equal 1%. Assume that the line-of-business risk and profitability parameters are the same as before.

The numerical results are summarized in Table 2. At the optimum asset allocation, 70.59% of assets are invested in Line 1 (versus 74.56% in Table 1). Asset risk is 9.19% (versus 9.03%). The overall capital ratio is 8.77% (versus 8.04%). Total assets are $11,397 (versus $12,439). The APV of the portfolio is $379 (versus $406). Thus, when the constraint is imposed on the total default value (instead of the default value per unit of liabilities), the allocation shifts towards the riskier assets with the higher gross margin. The capital ratio for the portfolio is higher as a result. With capital fixed, assets are lower and liabilities are higher. With the default value fixed, liabilities are safer: the default-to-liability ratio is 0.91% (versus 1.00%).

5. Implications for Regulation

Regulators do not allocate capital, but they do set risk-based capital requirements, which sounds like nearly the same thing. Therefore we consider whether a regulator could set risk-based capital requirements that (1) limit the size of the regulated firm's default put and (2) still allow the firm to operate at its portfolio optimum. We will refer to banks, although our conclusions apply to any financial firm subject to prudential regulation. We are *not* here concerned with "regulatory arbitrage," which is the substitution of risky for safer assets *within* a line of business. Regulatory arbitrage is a difficult but distinct problem.

We have discussed how a bank's headquarters could implement an optimal portfolio of lines of business. In principle a regulator could do the same thing, if it had complete information. Then the regulator could calculate the optimum portfolio and set capital requirements equal to the optimum capital allocations. But the bank will view the regulators' capital requirements as *constraints* on its optimization, not as *conditions* for achieving the bank's optimal portfolio. Faced with these constraints, the bank will resolve its portfolio problem and move away from the optimum. In order to prevent the bank from moving away, the regulator will have to stand ready to charge the bank at the margin for any increase in the value of its default put or deterioration of its credit quality. An example follows.

5.1 Example

Suppose the regulator has full information—it knows bank capital as well as the risks and profitability of all candidate lines of business. The regulator wants to enforce a constraint on the present value of the bank's default put. It determines the capital requirements by solving the optimization problem set out in the last section. If we use "hats" to indicate the regulatory capital requirements, the bank's total regulatory capital requirement $\hat{C}$ will be based on the line-of-business capital requirements $\hat{c}_i$, with $\hat{C} = \sum_i \hat{c}_i A_i$.

This outcome would implement risk-based capital regulation in its purest form, where custom requirements $\hat{c}_i$ are set for each bank, incorporating the bank's optimal portfolio and the joint probability distribution of returns on the bank's assets. But regulation can fundamentally change the bank's optimization problem, because the regulatory constraint $\sum_i \hat{c}_i A_i \le \overline{C}$ replaces the put value constraint.

In this case we can write the bank's optimization problem and optimality conditions as:

$$L(A_i, \lambda_c) = \sum_i A_i \left(1 - e_i - \tau \hat{c}_i\right) + \lambda_C \left(\overline{C} - \sum_i \hat{c}_i A_i\right) \qquad (31)$$

The conditions for an optimum are

(31a) $$\frac{\partial L}{\partial A_i} = 1 - e_i - \left(\tau + \lambda_C\right)\hat{c}_i = 0$$

(31b) $$\frac{\partial L}{\partial \lambda_C} = \overline{C} - \sum_i \hat{c}_i A_i = 0$$

The bank will calculate its APV as follows:

$$APV = \sum_i A_i \left(1 - e_i - \tau \hat{c}_i\right) = A \sum_i x_i (1 - e_i) - \tau \overline{C}, \tag{32}$$

where $x_i \equiv \frac{A_i}{A}$ are the proportional asset allocations. Because the bank's total capital requirement depends on its allocation of assets, $\hat{C} = \sum_i A x_i \hat{c}_i$, and it operates subject to a capital constraint, $\overline{C}$, total assets change when asset allocations change: $A = \frac{\overline{C}}{\hat{c}}$.

Suppose the regulator sets risk-based capital requirements $\hat{c}_i$ equal to the capital allocations in Table 2, that is, 7.34% for asset 1 and 12.23% for asset 2. If the bank's cost (if any) for credit backup is sunk and not affected by the risks it takes, then it will ignore default value. If the bank pushes its asset allocation towards lines of business with higher gross profitability, it will reduce the amount of assets in which it can invest. At the capital requirements for each line-of-business determined by the regulator, the ratio $\frac{1 - e_i}{c_i}$ is identical for all lines of business, i.e., higher profitability per unit of assets will be exactly offset by the reduction in assets the bank's capital can support. Since the regulatory capital requirements are fixed, bank profitability will be the same regardless of its business mix. Therefore the regulator's capital requirements are consistent with the bank's optimal portfolio, but the bank has no incentive to *stay* at that portfolio.

We can show, however, that these regulatory capital requirements lead the bank to its optimal portfolio if the bank bears a fraction ϕ of any change in the value of its default put. Incorporating this cost sharing rule in the optimization problem solved by the regulator does not affect the optimal capital allocations. If the term ϕP is added to the objective function (i.e., deducted from the bank's profit), the optimality conditions change only insofar as the term $\phi + \lambda_p$ replaces the term λ_p. Thus, the cost sharing parameter ϕ reduces the shadow price of the constraint on the default value one for one, so the capital allocation parameters are the same whether ϕ is zero or greater than zero.

In contrast, bearing a portion of the cost of the default put does change the solution to the bank's optimization problem. The optimization problem and optimality conditions become

$$L(A_i, \lambda_c) = \sum_i A_i (1 - e_i - \tau \hat{c}_i) - \phi P + \lambda_C \left(\overline{C} - \sum_i \hat{c}_i A_i \right) \qquad (34)$$

(34a) $$\frac{\partial L}{\partial A_i} = 1 - e_i - (\tau + \lambda_C)\hat{c}_i - \phi p_i = 0$$

(34b) $$\frac{\partial L}{\partial \lambda_C} = \overline{C} - \sum_i \hat{c}_i A_i = 0$$

Condition (34a) can be re-written as follows:

$$\frac{\partial L}{\partial A_i} = 1 - e_i - \hat{c}_i \left(\tau + \lambda_C + \phi \left(\frac{p_i}{\hat{c}_i} \right) \right) = 0 \qquad (35)$$

Recall that at an optimum the term $\tau + \lambda_C + \phi \left(\frac{p_i}{c_i} \right)$ is the same for all lines of business—it is the all-in cost of capital. Therefore, Eq. (35) becomes

$$\frac{\partial L}{\partial A_i} = 1 - e_i - \hat{c}_i \left(\tau + \lambda_C + \phi \left(\frac{p}{\hat{c}} \right) \right) = 0$$

An increase in ϕ will be offset by a decrease in the shadow price of capital. If the bank and the regulator have the same information about line-of-business risk and profitability, they will also share opinions about the all-in cost of capital regardless of the share ϕ of the default value the bank bears. The bank will therefore allocate assets consistent with the regulatory optimization problem.

Table 3 summarizes the numerical results for the bank when it faces line-of-business capital requirements $\hat{c}_1 = 7.34\%$ and $\hat{c}_2 = 12.23\%$. All other inputs are the same as in Table 2. Panel A shows the results when the bank bears none of the cost of the default put and Panel B the results when it bears 20% of the cost. The "default premium" is the amount the bank would have to pay to obtain full credit support. When the bank bears none of the cost of the default put at the margin, its profitability is the same at every possible asset allocation; asset allocation is irrelevant to its profitability.[23] In Panel B, where the bank bears a portion of the cost of the default option, its asset allocation decision matches the asset allocation in the optimization problem solved by the regulator.

We have derived similar results where the regulator wants to enforce a constraint on credit quality. If the regulator solves the bank's optimization problem subject to this constraint, and then uses the resulting capital allocations as line-of-business capital requirements, the bank will stay at its optimum portfolio only if the regulator penalizes the bank at the margin for any decrease in credit quality from the regulator's target. The penalty is a fraction of the product of the default-free debt value and the difference between actual and target credit quality: $\phi D\left(\frac{P}{D}-\alpha\right)$. Absent the penalty, the bank has no reason to stay at its optimum portfolio.

5.2 Policy implications

Of course these examples are too good to be true. No regulator implements custom risk-weighted capital requirements for each bank. If it tried to do so, it would not know enough about the bank to calculate the capital requirements accurately. The regulator would have to know the profitability of each of the bank's lines of business and the *joint* probability distribution of the lines' returns. For example, suppose the regulator understands the risk of the bank's candidate lines of business but not their profitability. Table 4 modifies Table 3 to show what happens when the regulator determines capital requirements based on the assumption that both lines of business have the same gross profitability ($e_1 = e_2$). The regulator sets capital requirements for both lines to 8.54%. The bank bears 20% of the cost of the default option (as in Table 3 Panel B). In this case the bank allocates all of its assets to Line 2 and its default value is $518, far above the regulators target default value of $95.

Perverse outcomes also arise if the regulator cannot assess the joint probability distribution of returns on a bank's lines of business or the tax or other costs of bank capital. Outcomes can be still worse if the regulator imposes standardized capital requirements on banks with different lines of business or different risks and returns in common lines of business.

To sum up, if the bank faces line-of-business capital requirements determined by a regulator with full information, and if it is penalized by the regulator if it acts to increase its default put or reduce its credit quality, then it will voluntarily choose its optimal portfolio and also meet the regulator's targets. But this ideal result is impossible

[23] This irrelevance result depends on the assumptions of constant returns to scale (constant 1- e_1 and 1- e_2). Decreasing returns to scale would prevent the bank from moving to corner solutions with only one line of

in practice. Therefore we must conclude that *bank regulation based on risk-weighted capital requirements will always distort a bank's decisions about its optimal portfolio of lines of business if capital is costly and the constraint on credit quality or default-put value is binding.* Moreover, regulators' ability to penalize banks at the margin for decisions that reduce credit quality or increase default-put values is in practice limited.[24] Therefore regulation by risk-based capital requirements will in many cases encourage risk-seeking portfolio choices.

If a regulator really had the information required to set ideal risk-based capital requirements for every one of a bank's lines of business – as in Panel B of Table 3, for example – the regulator would not need the line-by-line requirements. It would have sufficient information to enforce a constraint on the bank's *overall* credit quality or default-put value. The bank could then solve its optimal portfolio and capital allocations subject to the constraint, and would allocate capital efficiently. Risk-based capital requirements would be redundant.

Thus risk-based capital requirements are not needed in an ideal world and distorting in the real world. Temptations for regulatory arbitrage add another layer of difficulty, in addition to the problems set out here.

Perhaps risk-based capital requirements are "better than nothing;" it is hard for us to say without specifying realistic alternatives and undertaking an investigation far beyond the scope of this paper. But the built-in difficulties of capital regulation should enforce the case for more capital at banks and other financial institutions, for example as

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business. The firm would still move from the optimum, however.

[24] In order for the regulator to penalize banks for adverse marginal changes in portfolio composition, the regulator would have to distinguish declines in credit quality due to actions by the bank from declines from events outside the bank's control.

advocated by Admati et al. (2010) and Hellwig (2010). Other things equal, more capital reduces the value of the default put and dilutes the regulated firm's incentives to take advantage of regulatory requirements, which are necessarily imperfect.

6. Conclusions

We argue that capital can and should be allocated depending on the marginal default value of each line of business, where marginal default value is the derivative of the value of the bank's default put with respect to a change in the scale of the business. Capital allocations are relevant for pricing, performance measurement, incentives, compensation, and trading and hedging decisions.

Capital allocations based on marginal default values add up exactly. This adding-up result requires complete markets, complete enough that the bank's assets and default put option have well-defined market values, but does not require any restrictions on the joint probability distributions of returns. Allocations will be sensitive to distributional assumptions, however.

Differences in marginal default values across lines of business should be cancelled out by offsetting differences in marginal capital allocations. We calculate the resulting capital allocations from the conditions for an optimal portfolio of lines. The allocations are systematically different from allocations proportional to VaR or contribution VaR. For example, VaR measures allocate zero capital to risk-free assets. Our allocations are always negative for risk-free assets and often negative for low-risk assets.

Our results have implications for bank regulation. First, the correct charge for a one-period government guarantee of a bank's deposits and debt is the put value P. Capital requirements should not be based on the probability of failure, but on the present value of losses when and if failure occurs. This result repeats Merton (1977). Second, risk-weighted capital requirements should in principle not be based on VaR or contribution VaR. For example, the risk weight for safe assets (with VaR = 0) should be negative. Risk weights of zero for such assets encourage banks to shift investment away from safe and low-risk assets to riskier assets. Third – and probably most important -- it is exceptionally difficult to find risk-weighted capital requirements that work. By "work," we mean risk-weighted requirements that do *not* tempt the bank distort its optimal portfolio of businesses in order to expand its default put. If a regulator did have sufficient information to design "custom" capital requirements for each bank's lines of business, and if the regulator could penalize the bank for excessive risk-taking, then the capital requirements would not be needed. The regulator would have all the information required to set a floor for credit quality or a ceiling for the default put. If the floor or ceiling is enforced, then the regulator gains nothing by attempting to set capital requirements line by line.

We derive capital allocations from the conditions for an optimal portfolio of lines of business. Of course no bank or financial firm solves an explicit mathematical program to determine its optimal portfolio at the start of every period. Usually the firm takes its existing portfolio as fixed or considers gradual marginal changes. Our capital allocation procedures also work for any fixed portfolio, however.

Sometimes a bank or financial firm has to decide whether to add or subtract a line of business or a significant block of assets -- see Merton and Perold (1993) and Perold (1995). The decision hinges on whether the bank is better off with or without the business or assets. Capital allocations "with" are not the same as "without." All capital allocations can change after a discrete investment. The only general way to evaluate discrete changes is to compare value with vs. without, using different capital allocations.

If the discrete change is small relative to the bank's overall assets, allocations for existing lines can be a good approximation if the bank has many existing lines of business and if the line of business that is changed is not too large. Allocations for a business that is expanded or contracted can be very sensitive to the magnitude of the change, but allocations to existing businesses can be much more stable and for practical purposes may not have to be adjusted.[25]

Consider a proposal to add an entirely new business. The new business's NPV is reduced by the cost of the capital allocated to it. The investment is worthwhile if *net* NPV is positive, taking the mix of existing businesses as constant. Net NPV means NPV minus the all-in cost of allocated capital, which includes the tax or other costs of holding capital and the shadow price on the capital constraint.[26] The amount of capital allocated increases steadily as the scale of the new business increases. Thus capital allocation is a source of decreasing returns to investment.[27] Optimal scale (holding existing assets constant) is reached when NPV minus the all-in cost of allocated capital is zero.

[25] Myers and Read (2001) perform experiments showing that allocations to existing lines change slowly when new lines are added and subtracted. We have run similar experiments with similar results.

[26] If raising equity capital is feasible but incurs transaction costs, the marginal transaction costs should be charged against APV in place of the shadow price on the capital constraint.

[27] Line-by-line APVs could not be used to construct the optimal overall mix of business, however. The APV of each business would depend on the order in which candidate businesses were evaluated. This problem is highlighted by Merton and Perold (1993) and Perold (2005).

We have assumed tax costs of holding capital. Bank capital is also said to be costly because of agency and information costs. See Merton and Perold (1993) and Perold (2005), for example. These costs are less clear. For example, if the bank is not fully transparent, additional capital should add value, not reduce it, because banks do business with counterparties who depend on the bank's credit. If the number of counterparties is large, the total cost of counterparties' due diligence and continuing credit tracking of the bank can be significant. These costs are passed on to the bank as less favorable terms on the banks' transactions. A bank with more capital, other things equal, imposes lower costs on counterparties and should be more profitable. Thus lack of transparency is an argument for more capital, not less.

We have yet to see a good explanation for agency costs of bank capital. Are they costs of free cash flow, where managers are reluctant to curtail investment and release cash to shareholders? Adding debt in place of equity is regarded as a treatment or cure for this free-cash-flow problem. But too much debt could force managers to disinvest inefficiently *early*.[28] There is no reason to believe that more debt and less equity always add value. There is no reason to believe that more capital in a bank always generates more agency costs. For example, a cushion of extra capital can protect franchise value and forestall regulatory intervention if the bank suffers temporary losses. We believe that the agency costs of bank capital have to be thought through much more carefully.

[28] Lambrecht and Myers (2008) show how debt can lead to too much "discipline" and to *under*investment.

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Appendix 1

Observation 1 The default value P can be allocated uniquely across assets, proportional to the assets' marginal default values p_i:

$$P = \sum_{i=1}^{M} p_i A_i.$$

A proof of the observation for the two-asset case is below. Generalization to M assets is straightforward.

The amount of debt and deposits depends on A and a parameter c, the *capital ratio*, which measures the amount of capital that the bank puts up to back its liabilities. The capital ratio is a choice made by the bank or its regulators. For now we take c as constant across the bank's lines of business, with $D = (1-c)(A_1 + A_2)$. Therefore:

$$\begin{aligned} P &= \int_Z [R_D(1-c)(A_1 + A_2) - R_1 A_1 - R_2 A_2]\pi(z)dz \\ &= A_1 \int_Z [R_D(1-c) - R_1]\pi(z)dz + A_2 \int_Z [R_D(1-c) - R_2]\pi(z)dz. \end{aligned}$$

Changes in A_1 and A_2 affect limits of integration at the boundary of the default region Z. These marginal effects can be left out, however, because the put payoff on the boundary is zero.[29] Thus p_1 and p_2 are:

$$p_1 = \frac{\partial P}{\partial A_1} = \int_Z \pi(z)[R_D(1-c) - R_1]dz,$$

$$p_2 = \frac{\partial P}{\partial A_2} = \int_Z \pi(z)[R_D(1-c) - R_2]dz.$$

[29] Even if there were value effects from shifts of the boundary, we can show that the effects would cancel. Appendix 1 in Myers and Read (2001) shows how boundary changes cancel.

Multiply p_1 and p_2 by the respective asset values A_1 and A_2 to get the adding-up result, $p_1 A_1 + p_2 A_2 = P$.

Appendix 2.

This appendix provides detailed description of the variables presented in each row in Tables 1 and 2.

Asset allocation	Proportion of total assets invested in Line 1 ("x")
Asset risk	Standard deviation of returns on a portfolio with proportion x of assets invested in Line 1 & balance invested in Line 2
Capital ratio (in Table 1)	Ratio ("c") of capital ("C") to assets ("A") required to achieve minimum credit quality (default-to-liability ratio).
Capital ratio (in Table 2)	Ratio ("c") of capital ("C") to assets ("A")
Assets (in Table 1)	Value of total assets in portfolio (A = C / c)
Assets (in Table 2)	Value of total assets ("A") consistent with constraints on capital ("C" where C = 1,000) and default value ("P" where P = 95)
Line 1	Value of assets invested in Line 1 ("A_1" where A_1= x*A)
Line 2	Value of assets invested in Line 2 ("A_2" where A_2 = (1-x)*A)
Liabilities	Present value of default-free liabilities ("L" where L = A - C)
Default value	Present value of option to default ("P"); obtained via risk-neutral valuation under the assumption of normal return distributions
APV	Present value of portfolio after deducting the cost of risk capital (APV = NPV− τ*C)
All-in cost of capital	The sum of the market cost of capital (τ) and the internal shadow price of capital
Default-to-liability ratio	The present value of the default put expressed as a ratio to the present value of default-free liabilities
Default-to-asset ratio	The present value of the default put expressed as a ratio to the present value of assets
Default-to capital ratio	The present value of the default put expressed as a ratio to capital
Variance A	The variance of returns on the portfolio
Covariance 1,A	The covariance of returns on Line 1 with returns on the portfolio
Covariance 2,A	The covariance of returns on Line 2 with returns on the portfolio
Marg. default value Line 1	Marginal default value for Line 1 ("p_1") calculated using Eq. (27) for the case with constraint on default-to-liability ratio in Table 1 and for the case with constraint on default value in Table 2
Marg. default value Line 2	Marginal default value for Line 2 ("p_2") calculated using Eq. (27) for the case with constraint on default-to-liability ratio in Table 1 and for the case with constraint on default value in Table 2
Capital allocation Line 1	Capital allocation rate for Line 1 ("c_1") calculated using Eq. (29) in Table 1 and Eq. (30) in Table 2.
Capital allocation Line 2	Capital allocation rate for Line 2 ("c_2") calculated using Eq. (29) in Table 1 and Eq. (30) in Table 2.
Capital	Equal to the capital constraint (C = 1,000)
Line 1	Equal to the product of the Line 1 capital allocation rate and Line 1 assets ($C_1 = c_1*A_1$)
Line 2	Equal to the product of the Line 2 capital allocation rate and Line 2 assets ($C_2 = c_2*A_2$)
Capital charge Line 1	Equal to the product of the market cost of capital and Line 1 capital (= $\tau*C_1$)
Capital charge Line 2	Equal to the product of the market cost of capital and Line 2 capital (= $\tau*C_2$)
APV Line 1	Adjusted present value of Line 1 is net present value of Line1 less the cost of allocated capital ($APV_1 = NPV_1 - \tau*C_1$)
APV Line 2	Adjusted present value of Line 2 is net present value of Line1 less the cost of allocated capital ($APV_2 = NPV_2 - \tau*C_2$)
Marginal profit Line 1	Marginal profit reflects the all-in cost of capital (including shadow price)
Marginal profit Line 2	Marginal profit reflects the all-in cost of capital (including shadow price)

Table 1 - Capital Allocation with Constraint on Credit Quality

This table presents numerical examples for a firm selecting a two-line portfolio with a constraint on the credit quality (default-to-liability ratio). We assume that the firm has \$1,000 of capital and maintains a default-to-liability ratio of 1%. The standard deviations of asset returns for Lines 1 and 2 are 10% and 20%, respectively; and the returns are uncorrelated. The per-dollar net present values (gross margins) are 3% and 5%, respectively. The cost of capital is τ = 3%. See Appendix 2 for a detailed description of the variables in each row.

Asset allocation (Line 1)	74.56%	100%	90%	80%	70%	60%	50%	40%	30%	20%	10%	0%
Asset risk	9.03%	10.00%	9.22%	8.94%	9.22%	10.00%	11.18%	12.65%	14.32%	16.12%	18.03%	20.00%
Capital ratio	8.04%	9.55%	8.33%	7.91%	8.33%	9.55%	11.46%	13.95%	16.93%	20.31%	24.07%	28.16%
Assets	12,439	10,472	12,000	12,636	12,000	10,472	8,726	7,168	5,908	4,923	4,155	3,551
Line 1	9,275	10,472	10,800	10,109	8,400	6,283	4,363	2,867	1,773	985	416	0
Line 2	3,165	0	1,200	2,527	3,600	4,189	4,363	4,301	4,136	3,938	3,740	3,551
Liabilities	11,439	9,472	11,000	11,636	11,000	9,472	7,726	6,168	4,908	3,923	3,155	2,551
Default value	115	95	110	117	110	95	77	62	49	39	32	26
APV	406	284	354	400	402	368	319	271	230	196	169	148
All-in cost of capital	44%	31%	38%	43%	43%	40%	35%	30%	26%	23%	20%	18%
Default-to-liability ratio	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%
Default-to-asset ratio	0.92%	0.91%	0.92%	0.92%	0.92%	0.91%	0.89%	0.86%	0.83%	0.80%	0.76%	0.72%
Default-to capital ratio	11.47%	9.50%	11.03%	11.67%	11.03%	9.50%	7.75%	6.19%	4.92%	3.93%	3.16%	2.56%
Variance A	0.0081	0.0100	0.0085	0.0080	0.0085	0.0100	0.0125	0.0160	0.0205	0.0260	0.0325	0.0400
Covariance 1,A	0.0075	0.0100	0.0090	0.0080	0.0070	0.0060	0.0050	0.0040	0.0030	0.0020	0.0010	0.0000
Covariance 2,A	0.0102	0.0000	0.0040	0.0080	0.0120	0.0160	0.0200	0.0240	0.0280	0.0320	0.0360	0.0400
Marg. default value Line 1	0.93%	0.91%	0.91%	0.92%	0.94%	0.97%	1.00%	1.03%	1.06%	1.09%	1.12%	1.15%
Marg. default value Line 2	0.89%	1.07%	0.99%	0.92%	0.86%	0.81%	0.78%	0.75%	0.74%	0.73%	0.72%	0.72%
Capital allocation Line 1	6.87%	9.55%	9.16%	7.91%	5.84%	3.22%	0.37%	-2.53%	-5.41%	-8.30%	-11.27%	-14.42%
Capital allocation Line 2	11.46%	-6.28%	0.85%	7.91%	14.15%	19.04%	22.55%	24.94%	26.50%	27.47%	27.99%	28.16%
Capital	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000
Line 1	637	1,000	990	800	491	202	16	-73	-96	-82	-47	0
Line 2	363	0	10	200	509	798	984	1,073	1,096	1,082	1,047	1,000
Capital charge Line 1	19	30	30	24	15	6	0	-2	-3	-2	-1	0
Capital charge Line 2	11	0	0	6	15	24	30	32	33	32	31	30
APV Line 1	259	284	294	279	237	182	130	88	56	32	14	0
APV Line 2	147	0	60	120	165	186	189	183	174	164	156	148
Marginal profit Line 1	0.00%	0.00%	-0.52%	-0.40%	0.48%	1.72%	2.87%	3.76%	4.41%	4.88%	5.25%	5.56%
Marginal profit Line 2	0.00%	6.97%	4.67%	1.60%	-1.11%	-2.58%	-2.87%	-2.51%	-1.89%	-1.22%	-0.58%	0.00%

Table 2 - Capital Allocation with Constraint on Default Value

This table presents numerical examples for a firm selecting a two-line portfolio with a constraint on the default value. We assume that the firm has $1,000 of capital and maintains a default value of $95. The standard deviations of asset returns for Lines 1 and 2 are 10% and 20%, respectively; and the returns are uncorrelated. The per-dollar net present values (gross margins) are 3% and 5%, respectively. The cost of capital is τ = 3%. See Appendix 2 for a detailed description of the variables in each row.

Asset allocation (Line 1)	70.59%	100%	90%	80%	70%	60%	50%	40%	30%	20%	10%	0%
Asset risk	9.19%	10.00%	9.22%	8.94%	9.22%	10.00%	11.18%	12.65%	14.32%	16.12%	18.03%	20.00%
Assets	11,397	10,472	11,359	11,708	11,359	10,472	9,367	8,279	7,314	6,495	5,809	5,236
Line 1	8,045	10,472	10,223	9,367	7,951	6,283	4,683	3,312	2,194	1,299	581	0
Line 2	3,352	0	1,136	2,342	3,408	4,189	4,683	4,967	5,120	5,196	5,228	5,236
Liabilities	10,397	9,472	10,359	10,708	10,359	9,472	8,367	7,279	6,314	5,495	4,809	4,236
Default value	95	95	95	95	95	95	95	95	95	95	95	95
APV	379	284	333	368	379	368	345	318	292	269	249	232
Capital ratio	8.77%	9.55%	8.80%	8.54%	8.80%	9.55%	10.68%	12.08%	13.67%	15.40%	17.22%	19.10%
All-in cost of capital	41%	31%	36%	40%	41%	40%	37%	35%	32%	30%	28%	26%
Default-to-liability ratio	0.91%	1.00%	0.92%	0.89%	0.92%	1.00%	1.14%	1.31%	1.50%	1.73%	1.98%	2.24%
Default-to-asset ratio	0.83%	0.91%	0.84%	0.81%	0.84%	0.91%	1.01%	1.15%	1.30%	1.46%	1.64%	1.81%
Default-to-capital ratio	9.50%	9.50%	9.50%	9.50%	9.50%	9.50%	9.50%	9.50%	9.50%	9.50%	9.50%	9.50%
Variance A	0.0084	0.0100	0.0085	0.0080	0.0085	0.0100	0.0125	0.0160	0.0205	0.0260	0.0325	0.0400
Covariance 1,A	0.0071	0.0100	0.0090	0.0080	0.0070	0.0060	0.0050	0.0040	0.0030	0.0020	0.0010	0.0000
Covariance 2,A	0.0118	0.0000	0.0040	0.0080	0.0120	0.0160	0.0200	0.0240	0.0280	0.0320	0.0360	0.0400
Marginal default value Line 1	0.70%	0.91%	0.89%	0.81%	0.69%	0.54%	0.41%	0.29%	0.19%	0.11%	0.05%	0.00%
Marginal default value Line 2	1.16%	0.00%	0.39%	0.81%	1.18%	1.45%	1.62%	1.72%	1.77%	1.80%	1.81%	1.81%
Capital allocation rate Line 1	7.34%	9.55%	9.32%	8.54%	7.25%	5.73%	4.27%	3.02%	2.00%	1.18%	0.53%	0.00%
Capital allocation rate Line 2	12.23%	0.00%	4.14%	8.54%	12.43%	15.28%	17.08%	18.12%	18.67%	18.95%	19.07%	19.10%
Capital	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000
Line 1	590	1,000	953	800	576	360	200	100	44	15	3	0
Line 2	410	0	47	200	424	640	800	900	956	985	997	1,000
Capital charge Line 1	18	30	29	24	17	11	6	3	1	0	0	0
Capital charge Line 2	12	0	1	6	13	19	24	27	29	30	30	30
APV Line 1	224	284	278	257	221	178	134	96	65	39	17	0
APV Line 2	155	0	55	111	158	190	210	221	227	230	231	232
Marginal profit Line 1	0.00%	0.00%	-0.39%	-0.40%	0.04%	0.72%	1.40%	1.95%	2.36%	2.65%	2.85%	3.00%
Marginal profit Line 2	0.00%	5.00%	3.49%	1.60%	-0.08%	-1.08%	-1.40%	-1.30%	-1.01%	-0.66%	-0.32%	0.00%

Table 3 - Bank Profitability with Line-of-Business Capital Requirements

This table presents numerical results for the bank when it faces line-of-business capital requirements of 7.34% for Line 1 and 12.23% for Line 2. We assume that the firm has $1,000 of capital. The standard deviations of asset returns for Lines 1 and 2 are 10% and 20%, respectively; and the returns are uncorrelated. The per-dollar net present values (gross margins) are 3% and 5%, respectively. The cost of capital is τ = 3%.

Asset allocation (Line 1)	70.59%	100%	90%	80%	70%	60%	50%	40%	30%	20%	10%	0%
Asset risk	9.19%	10.00%	9.22%	8.94%	9.22%	10.00%	11.18%	12.65%	14.32%	16.12%	18.03%	20.00%
Assets	11,397	13,632	12,780	12,028	11,360	10,762	10,224	9,737	9,294	8,890	8,520	8,179
Line 1	8,045	13,632	11,502	9,622	7,952	6,457	5,112	3,895	2,788	1,778	852	0
Line 2	3,352	0	1,278	2,406	3,408	4,305	5,112	5,842	6,506	7,112	7,668	8,179
Liabilities	10,397	12,632	11,780	11,028	10,360	9,762	9,224	8,737	8,294	7,890	7,520	7,179
Default value	95	184	130	102	95	102	120	145	174	206	238	271
Capital ratio	8.77%	7.34%	7.82%	8.31%	8.80%	9.29%	9.78%	10.27%	10.76%	11.25%	11.74%	12.23%
Panel A: Bank Share of Cost of Default Put = 0%												
Default premium	0	0	0	0	0	0	0	0	0	0	0	0
APV	109	109	109	109	109	109	109	109	109	109	109	109
Panel B: Bank Share of Cost of Default Put = 20%												
Default premium	19	37	26	20	19	20	24	29	35	41	48	54
APV	90	72	83	88	90	88	85	80	74	68	61	55

Table 4 - Bank Profitability with Same Capital Requirements across Lines of Business

This table presents numerical results for the bank when it faces line-of-business capital requirements of 8.54% for both lines. We assume that the firm has $1,000 of capital. The standard deviations of asset returns for Lines 1 and 2 are 10% and 20%, respectively; and the returns are uncorrelated. The per-dollar net present values (gross margins) are 3% and 5%, respectively. The cost of capital is τ = 3%.

Asset allocation (Line 1)	0%	49.98%	65.73%	100%	90%	80%	70%	60%	50%	40%	30%	20%	10%	0%
Asset risk	20.00%	11.18%	9.50%	10.00%	9.22%	8.94%	9.22%	10.00%	11.18%	12.65%	14.32%	16.12%	18.03%	20.00%
Assets	11,710	11,710	11,710	11,710	11,710	11,710	11,710	11,710	11,710	11,710	11,710	11,710	11,710	11,710
Line 1	11,710	5,853	7,697	11,710	10,539	9,368	8,197	7,026	5,855	4,684	3,513	2,342	1,171	0
Line 2	0	5,857	4,012	0	1,171	2,342	3,513	4,684	5,855	7,026	8,197	9,368	10,539	11,710
Liabilities	10,710	10,710	10,710	10,710	10,710	10,710	10,710	10,710	10,710	10,710	10,710	10,710	10,710	10,710
Capital ratio	8.54%	8.54%	8.54%	8.54%	8.54%	8.54%	8.54%	8.54%	8.54%	8.54%	8.54%	8.54%	8.54%	8.54%
Panel A: Bank Share of Cost of Default Put = 20%														
Default value	518			128	103	95	103	128	168	221	284	356	435	518
Default premium	104			26	21	19	21	26	34	44	57	71	87	104
APV	452			296	324	349	371	389	405	418	428	437	445	452
Panel B: Bank Share of Cost of Default Put = 50%														
Default value		168		128	103	95	103	128	168	221	284	356	435	518
Default premium		84		64	52	48	52	64	84	110	142	178	217	259
APV		355		257	293	321	340	351	355	351	343	330	315	296
Panel C: Bank Share of Cost of Default Put = 100%														
Default value			112	128	103	95	103	128	168	221	284	356	435	518
Default premium			112	128	103	95	103	128	168	221	284	356	435	518
APV			290	193	241	273	288	287	271	241	201	152	97	37