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Complex Times: Asset Pricing and Conditional Moments under Non-Affine Diffusions*

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Abstract

Many applications in continuous-time financial economics require calculation of conditional moments or contingent claims prices, but such expressions are known in closed-form for only a few specific models. Power series (in the time variable) for these quantities are easily derived, but often fail to converge, even for very short time horizons. We characterize a large class of continuous-time non-affine conditional moment and contingent claim pricing problems with solutions that are analytic in the time variable, and that therefore can be represented by convergent power series. The ability to approximate solutions accurately and in closed-form simplifies the estimation of latent variable models, since the state vector must be extracted from observed quantities for many different parameter vectors during a typical estimation procedure.

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1. Introduction

Many applications in economics and finance require solutions to second order parabolic partial differential equations with a final condition. Continuous-time processes are often expressed as solutions to stochastic differential equations; estimation of the parameters of such a model can be performed by a variety of techniques, including maximum likelihood or method of moments. Likelihood functions solve the Chapman-Kolmogorov forward and backward equations, whereas conditional moments solve the backward equation. Prices of derivative securities with European-style exercise are solutions to the Feynman-Kac equation with a final condition. In term structure models, bonds are often treated as derivatives written on the interest rate, and are therefore also solutions to the Feynman-Kac equation. In some estimation problems, both equations are encountered. For example, a model may be written in terms of a set of latent variables. In this case, the values of the state variables must be inferred from security prices or other observed quantities by inverting the Feynman-Kac solution, and the fit must be evaluated by calculating the likelihood function or conditional moments, using the Chapman-Kolmogorov equations.

However, the class of continuous-time models with closed-form conditional moments, likelihood functions, or derivative prices is quite limited. For the geometric Brownian motion model of equity prices used by Black and Scholes (1973) and Merton (1973), likelihoods, conditional moments of the state variable, and prices of standard derivative securities are all known in closed-form. In the term structure models of Vasicek (1977) and Cox, Ingersoll, and Ross (1985), likelihoods, conditional moments, and bond prices are all known in closed-form.[1] However, more complicated models almost always lose some of the tractability of these early models; for example, Heston (1993) uses Fourier transforms to find option prices in the stochastic volatility model of Hull and White (1987). In the general affine yield models of Duffie and Kan (1996), conditional moments of the state variables are known in closed-form, but (except for a few special cases) neither bond prices nor likelihoods can be found explicitly, so some numeric procedure is needed.[2] Nonetheless, much research on the term structure of interest rates has focused on affine yield models, since, for these models, the numeric procedure to calculate bond prices is very fast.[3] Estimation for this class of models has been by simulated method of moments, from Dai and Singleton (2000), by quasi-maximum likelihood, from Duffee (2002), and by closed-form approximation to likelihoods, as in Thompson (2004), Mosburger and Schneider (2005), and

[1] Our usage of "closed-form" includes such expressions as the cumulative Gaussian distribution function and modified Bessel functions of the first kind. With a narrower definition of "closed-form," the class of models for which such closed-form solutions exist is even more limited.

[2] Although partial differential equation techniques have been used to price bonds (and other interest rate derivative securities) under affine term structure models for some time, theoretical justification of this practice for the full class of affine models was not provided until recently; see Levendorskii (2004a) and Levendorskii (2004b). More information about affine yield models may be found in Dai and Singleton (2000), who develop a classification scheme, Gouriéroux and Sufana (2006), who show that there exist some affine models that lie outside of this classification scheme, Dai and Singleton (2002), who examine expectations puzzles in the context of affine models, and Duffee (2002), Duarte (2004), and Cheridito, Filipović, and Kimmel (2007), who explore alternative market price of risk specifications.

[3] In a general multiple factor term structure model, numeric solution of the partial differential equation that bond prices satisfy is typically very slow. However, if the coefficients of the PDE are linear, as they are in affine yield models, then the equation is equivalent to a system of Ricatti-type ODEs, which can be solved numerically very quickly.

Cheridito, Filipović, and Kimmel (2007).

Non-linear models are much less common in the literature, despite evidence from, for example, Aït-Sahalia (1996) and Stanton (1997) of non-linear evolution of the short interest rate process. For some classes of non-linear term structure models, such as Beaglehole and Tenney (1992), Constantinides (1992), Ahn and Gao (1999), and Ahn, Dittmar, and Gallant (2002), bond prices are known in closed-form. Grasselli and Tebaldi (2004) examine a class of term structure models with closed-form bond prices; this class includes affine yield models, but some other models as well. Duarte (2004) constructs a non-linear model that becomes linear under risk-neutral probabilities. Chan, Karolyi, Longstaff, and Sanders (1992) estimate a strongly non-linear model of the interest rate, but do not derive bond prices. In general, the numeric analysis required of many non-linear models makes their use difficult or impossible for many applications. When the state variables of a model are directly observed, or when they can be extracted from observed prices through closed-form expressions or fast numeric methods, techniques such as that of Aït-Sahalia (2002) and Aït-Sahalia (2008), who constructs a series of approximations to the likelihood function of a non-affine diffusion, can be used. Cheridito, Filipović, and Kimmel (2007), Thompson (2004), and Mosburger and Schneider (2005) apply this approach to affine yield models. In principle, this method of approximation can be extended to general solutions to the Feynman-Kac equation (such as bond prices in a term structure model) by integrating over the fundamental solution, which, in the special case of the Chapman-Kolmogorov backward equation, is the likelihood function. But apart from the integration, there are problems with this approach; the power series approximations may not converge at all for bonds with longer maturities; even if they do, such convergence is often so slow as to make their use impractical. Thus, despite recent advances in the estimation of non-linear models, significant challenges remain in estimation when the values of latent state variables must be extracted from observed prices.

We therefore develop a technique for the construction of series approximations to solutions of the Chapman-Kolmogorov backward and Feynman-Kac equations, which, as discussed, are conditional moments and contingent claim prices. Specifically, we use power series in the time variable. The conditional moment sought (or the final payoff of the contingent claim being priced) specifies the first coefficient in the power series; the Chapman-Kolmogorov and Feynman-Kac equations can then be used to establish a recursive relation, in which each coefficient beyond the first can be found by applying a functional to the previous coefficient. Applying this approach to a large class of scalar diffusion processes (and also multiple diffusions, provided the state variables evolve independently), we construct a large class of final conditions such that the corresponding moments are analytic in the time variable, and also a large class of non-affine term structure models for which bond prices are analytic in maturity. Analyticity in the time variable is important, since it is both a necessary and sufficient condition for convergence of the power series representation of the solution. Furthermore, the method of time transformations, as described in Kimmel (2008b), can often greatly increase the range of maturities for which the series converge (and also the speed of convergence). Our technique is then suitable for bond pricing applications, in which we must often consider time horizons of many years; see Kimmel (2008a) and Jarrow, Li, Liu, and Wu (2006) for applications to the pricing of non-callable and callable bonds, respectively.

Throughout, we focus on *complex times* motivated by *real applications*. That is, although a conditional moment or bond pricing function has meaning only for positive real time horizons, the behavior of such functions for all complex values of the time variable determines whether a power series converges, and what the region of convergence is. It is necessary to determine, for example, whether a bond price function has singularities for negative or complex values of time-to-maturity; even though the practical problem is inevitably about positive and real values of time-to-maturity, bad behavior of the price function for negative or complex times (which are not meaningful in terms of the application) can prevent convergence of a power series for positive real times (which we do care about in applications). Throughout, we therefore take the perspective that conditional moments and bond prices are complex functions of a complex time argument, even though these quantities really only deserve to be called “moments” or “prices” for positive real values of the time argument.

The rest of this paper is organized as follows. In Section 2, we discuss the general problem of constructing series representations to solutions of conditional moments or contingent claim pricing problems, and illustrate some of the problems with this approach. In Section 3, we show that, for an arbitrary scalar diffusion process and interest rate specification, there exists an infinite-dimensional family of conditional moment and diffusion problems with solutions that are analytic, and which therefore have convergent power series representations. In Section 4, we explicitly characterize two large families of contingent claim and conditional moment problems with analytic solutions, and determine the range of convergence of the power series representations of the solutions. Section 5 illustrates our technique with examples motivated by bond pricing problems. In some examples, bond prices are known in closed-form, allowing us to assess the accuracy of the approximations; in others, bond prices are not known in closed-form, but can nonetheless be approximated by our technique. Finally, Section 6 concludes. Proofs of all theorems and corollaries are found in the appendix, which also includes some auxiliary lemmas not shown in the main text.

2. Series Solutions

We consider an N-dimensional diffusion process:

$$X_{t+\Delta} = X_t + \int_t^{t+\Delta} \mu\left(X_u\right) du + \int_t^{t+\Delta} \sigma\left(X_u\right) dW_u$$

with initial condition $X_t = x$, where W_t is an N-dimensional standard Brownian motion, X_t is an N-vector of state variables, $\mu\left(X_t\right)$ is an $N \times 1$ vector-valued function, and $\sigma\left(X_t\right)$ is an $N \times N$ matrix-valued function. We assume that $\mu\left(X_t\right)$ and $\sigma\left(X_t\right)$ are chosen so that a unique strong solution X_t exists. There are many criteria for existence and uniqueness of solutions to stochastic differential equations in the literature; see, for example, Karatzas and Shreve (1991), Stroock and Varadhan (1979), or Liptser and Shiryaev (2001). We do not specify the particular existence and uniqueness requirements imposed, so as not to result in a loss of generality. We are interested in finding expectations of the following form, conditional on knowledge of the state vector at an earlier time:

$$f\left(\Delta, x\right) = E\left[e^{-\int_t^{t+\Delta} r\left(X_u\right) du} g\left(X_{t+\Delta}\right) \middle| X_t = x\right] \tag{2.1}$$

for some scalar-valued functions $r(x)$ and $g(x)$, and for some time horizon $\Delta \geq 0$.[4] Note that we do not specify whether the expectation is to be calculated under true or risk-neutral probabilities (or under some other artificial probability measure, such as a risk-forward measure). For conditional moment problems, the expectation in (2.1) is usually taken under true probabilities (with $r(x) = 0$), whereas for asset pricing problems, the expectation is taken under risk-neutral or risk-forward probabilities. Under technical regularity conditions,[5] the solution $f(\Delta, x)$ to the probabilistic problem is also a solution to the partial differential equation:

$$\frac{\partial f}{\partial \Delta}(\Delta, x) = \sum_{i=1}^{N} \mu_i(x) \frac{\partial f}{\partial x_i}(\Delta, x) + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij}^2(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(\Delta, x) - r(x) f(\Delta, x) \tag{2.2}$$

with the final condition $f(0, x) = g(x)$, where $\mu_i(x)$ denotes the ith element of the vector $\mu(x)$, and $\sigma_{ij}^2(x)$ denotes the element in the ith row and jth column (or, by symmetry, the jth row and ith column) of the matrix $\sigma(x)\sigma^T(x)$. A solution of (2.1) and (2.2) is the price of a derivative instrument with final payoff $g(X_t)$ at maturity. The Chapman-Kolmogorov backward equation is obtained by setting $r(x) = 0$; in this case, solutions to the partial differential equation are conditional expectations (also subject to technical regularity conditions).

Solutions to (2.2) are known in closed-form only for a few special cases. Approximations to conditional likelihood functions have been developed by Aït-Sahalia (2002) for scalar diffusions; see Aït-Sahalia (1999) for examples. This technique was extended to the case of multiple diffusions by Aït-Sahalia (2008). Since a conditional moment is an integral of the final condition over the likelihood function, it might seem this approach could be used to approximate solutions to (2.2) as well, at least in the case $r(x) = 0$. However, this approach is problematic. Consider a convergent series of approximations to a likelihood function:

$$\rho_n(\Delta, x, y) \Longrightarrow \rho(\Delta, x, y) \tag{2.3}$$

where x is the backward state variable, y is the forward state variable, and Δ is the time between the backward and forward observations. It does not necessarily follow that:

$$\int_{-\infty}^{+\infty} \rho_n(\Delta, x, y) g(y) \, dy \Longrightarrow \int_{-\infty}^{+\infty} \rho(\Delta, x, y) g(y) \, dy \tag{2.4}$$

Note that we have not specified the type of convergence in (2.3), e. g., point-wise or uniform. However, even if this convergence is uniform in y, there is no guarantee of any meaningful kind of convergence in (2.4), nor even the existence of the integrals on the left-hand side. Furthermore, even in cases where the conditional moment approximations do converge, it may be difficult or impossible to calculate these integrals explicitly. Finally, even if these difficulties can be overcome, the approximation method still must be extended to take the $r(x)$ coefficient in (2.2) into account, for approximation of contingent claims prices.

[4] This approach is a very typical one for pricing of contingent claims, such as options, or bonds in a term structure model. For a very different approach to modeling the term structure of interest rates, see Heath, Jarrow, and Morton (1992).

[5] See Levendorskii (2004a) and Levendorskii (2004b) for a recent discussion. General conditions for the equivalence of the probabilistic and partial differential equation problems that are necessary, sufficient, and simple to apply remain elusive.

We therefore take a different approach, which is to construct a power series representation to the conditional moment or contingent claims price directly, without going through the intermediate step of constructing a likelihood representation or fundamental solution. The form of the partial differential equation (2.2) suggests that the solution can be written as a power series in Δ, centered at zero:

$$f(\Delta, x) = a_0(x) + \sum_{n=1}^{\infty} a_n(x) \frac{\Delta^n}{n!} \tag{2.5}$$

Since any power series representation of the solution to (2.2) converges in a region $|\Delta| < r$ for some $r \geq 0$, the final condition requires:

$$a_0(x) = g(x) \tag{2.6}$$

Substituting the proposed solution into (2.2), and gathering terms of like order in Δ, one finds that the functions $a_n(x)$ for $n \geq 1$ must satisfy a recursive relation:

$$a_n(x) = \sum_{i=1}^{N} \mu_i(x) \frac{\partial a_{n-1}}{\partial x_i}(x) + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij}^2(x) \frac{\partial^2 a_{n-1}}{\partial x_i \partial x_j}(x) - r(x) a_{n-1}(x) \tag{2.7}$$

Provided $g(x)$, $\mu(x)$, $\sigma(x)\sigma^T(x)$, and $r(x)$ are all infinitely differentiable in a neighborhood of x, the coefficients as defined above all exist.[6] The series described in (2.5), (2.6), and (2.7) can be interpreted as the deterministic part of the stochastic Itô-Taylor expansions as discussed in, for example, Kloeden and Platen (1999).

Given the requisite smoothness conditions of the three coefficients and the final condition, derivation of a power series representation of a solution of (2.2) is straightforward. Much less straightforward is determining where the series converges. Any power series converges trivially at the point where the series is centered, since all terms but the first are zero. However, for large values of Δ (and possibly for any $\Delta \neq 0$), the proposed power series solution may not converge; worse still, it may converge to the wrong function.

The probabilistic problem (2.1) is meaningful only for non-negative real values of the time horizon, $\Delta \in [0, +\infty)$. The partial differential equation problem (2.2) (with final condition) is motivated by the probabilistic problem, but it is nonetheless possible to consider solutions to the PDE problem which are defined for other values of Δ, for example, negative or imaginary values. When calculating power series representations, it is advantageous to consider the PDE solution in this more general setting, because, even though the solution has no meaning (in the context of the original probabilistic problem) for values of $\Delta \notin [0, +\infty)$, the behavior of the solution for complex values of Δ affects the region of the convergence of the power series. Even when the coefficients of the partial differential equation and the final condition satisfy strong smoothness conditions, a power series may fail to converge for some non-negative time horizons. To illustrate some of the problems

[6] Infinite differentiability of these functions is sufficient, but not necessary, for the existence of the coefficients $a_n(x)$. For example, if $g(x)$ and $\mu(x)$ are both affine in x, then the coefficients $a_n(x)$ can be found even if $\sigma(x)\sigma^T(x)$ is not differentiable. The coefficients $a_n(x)$ can even be found in some cases in which the coefficients of the partial differential equation do not specify a valid diffusion process in the analogous probabilistic problem. However, existence of the power series coefficients does not guarantee convergence of the series anywhere but the origin.

that can occur, we consider the very simple special case of finding conditional moments of a function of the terminal value of a Brownian motion. We seek the conditional moment:

$$f(\Delta, x) = E\left[\exp\left(\frac{cX_{t+\Delta}^2}{2}\right)|X_t = x\right]$$

where X_t is a Brownian motion. In this case, the coefficients of (2.2) are $\mu(x) = 0$, $\sigma(x) = 1$, and $r(x) = 0$, and the final condition is $g(x) = \exp\left(cx^2/2\right)$. The solution is:

$$f(\Delta, x) = \frac{\exp\left(\frac{cx^2}{2(1-\Delta c)}\right)}{\sqrt{1-\Delta c}}$$

Power series converge within a circle extending to the nearest singularity in any direction in the complex plane. This function has a singularity at $\Delta = 1/c$, and a power series about $\Delta = 0$ therefore converges only for $|\Delta| < 1/|c|$ (and possibly also for some points $|\Delta| = 1/|c|$). If $c > 0$, the conditional expectation is not defined for $\Delta \geq 1/c$, so non-convergence is appropriate for these values; the tails of the final condition grow too quickly as a function of x, and the conditional expectation becomes undefined for values of Δ that are too large.

However, for $c < 0$, the conditional moment is defined and satisfies strong smoothness conditions for all $\Delta \geq 0$. Nonetheless, the singularity at $\Delta = 1/c$ prevents convergence of the series for $\Delta > 1/|c|$, even though the conditional moment function is perfectly well-behaved in this range. In the $c > 0$ case, the series fails to converge for $\Delta > 1/c$ because the tails of the final condition grow too quickly; however, in the $c < 0$ case, the series fails to converge for $\Delta > 1/|c|$ because the tails go to *zero* too quickly. The conditional moment function exists and is well-behaved for $\Delta > 0$, but the power series representation fails to converge because of the behavior of the extension of the conditional moment function to $\Delta < 0$.

Excessively thick or thin tails in the final condition are not the only problems that can cause power series to fail to converge. Consider the conditional moment:

$$f(\Delta, x) = E\left[\cos\left(\frac{cX_{t+\Delta}^2}{2}\right)|X_t = x\right]$$

for any real $c \neq 0$, where X_t is (as before) a Brownian motion. (Since the cosine function is even, we can take $c > 0$ without loss of generality.) The solution is given by:

$$f(\Delta, x) = \frac{\exp\left(-\frac{x^2\Delta c^2}{2(1+\Delta^2c^2)}\right)}{\sqrt[4]{1+c^2\Delta^2}}\cos\left[\frac{cx^2}{2\left(1+c^2\Delta^2\right)} + \frac{\arctan(c\Delta)}{2}\right]$$

For positive values of Δ (i. e., the values that are meaningful in the probabilistic problem), we take the fourth root to be the positive branch. This solution is then well-behaved for all real values of Δ; however, there are singularities in $f(\Delta, x)$ for imaginary values of Δ, at $\Delta = \pm\imath/c$. As in the previous example, this power series converges for all $|\Delta| < 1/c$ (and possibly also for some $|\Delta| = 1/c$), but diverges elsewhere. Even though $f(\Delta, x)$ is well-behaved for *all* real values of Δ, singularities at imaginary values of Δ prevent convergence of a power series approximation for many real values. Here, the problem is not that the final condition goes either to infinity or to zero too quickly, but that it oscillates too rapidly in the tails.

Some power series fail to converge for any values except $\Delta = 0$. Consider the contingent claim price:

$$f(\Delta, x) = E\left[e^{-\int_t^{t+\Delta} r du} \max(X_{t+\Delta} - K, 0)\right]$$

where X_t is a geometric Brownian motion. The coefficients of (2.2) are then $\mu(x) = \mu x$, $\sigma(x) = \sigma x$, and $r(x) = r$, and the final condition is $g(x) = \max(x - K, 0)$. The solution is the well-known option pricing formula of Black and Scholes (1973) and Merton (1973):

$$f(\Delta, x) = X_t N\left(\frac{\ln\frac{X_t}{K} + \left(r + \frac{\sigma^2}{2}\right)\Delta}{\sigma\sqrt{\Delta}}\right) - Ke^{-r\Delta} N\left(\frac{\ln\frac{X_t}{K} + \left(r - \frac{\sigma^2}{2}\right)\Delta}{\sigma\sqrt{\Delta}}\right) \tag{2.8}$$

where $N(\bullet)$ is the cumulative normal distribution function. This solution is analytic in a neighborhood of any value of Δ *except* $\Delta = 0$.[7] Since the power series constructed as in (2.6) and (2.7) are centered at $\Delta = 0$, a series representation of (2.8) converges *only* for this value.

Of course, other than for their use as illustrative examples, there is little point in finding power series representations of functions that are already known in closed-form. However, the problems encountered in the examples above can also occur in those cases for which the solutions are not known in closed-form. Even if a conditional moment or asset price function is well-behaved for positive real values of Δ (i. e., those values of interest in typical applications), a power series fails to converge if the final (or payoff) condition has tails that, for example, are too thin, or oscillate too quickly. In these cases, singularities for negative or complex values of Δ prevent convergence of the series for positive real values of Δ. The next section considers the problem of determining when solutions to conditional moment or contingent claims pricing problems have analytic (in the time variable) solutions.

3. Existence of Analytic Solutions

It may not be obvious, for some choices of the $\mu(x)$, $\sigma(x)$, and $r(x)$ functions, that there are any final conditions $g(x)$ at all for which there exists a solution $f(\Delta, x)$ of (2.2) that is analytic in the time variable in some neighborhood of the origin. Analyticity of solutions for Δ with positive real part follows more or less directly from well-known results in the literature. For example, analyticity of the fundamental solution to a general problem on a bounded domain follows from the construction of Friedman (1964). Colton (1979) shows that, on a bounded domain, there exists, for any final condition, an approximate solution to the general scalar PDE problem, which is analytic in the time variable in a neighborhood of the origin.[8] However, these results are not useful for our purposes. The construction of Friedman (1964) does nothing to establish analyticity of the solution at $\Delta = 0$, which is necessary for convergence of a power series around that point; furthermore, these results apply to a bounded domain, not the unbounded domains typical in economic and financial

[7]The cumulative normal function may be extended to complex values of the argument by analytic extension, using its power series representation. This function is everywhere analytic, so the extension is unique.

[8]Colton (1979) focuses only on the backward heat equation, that is, negative values of the time variable, and although the solutions constructed are analytic in the time variable, this property is not noted in the paper.

applications. The results of Colton (1979) also apply to bounded domains, and furthermore, although the existence of an approximate solution is demonstrated, no practical method to find it is given. We therefore analyze the scalar case, and show that there exists an infinite-dimensional family of $g(x)$ that give rise to analytic solutions in a neighborhood of the origin, on domains that are not necessarily bounded. We first consider the special case:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{\sigma^2(y)}{2} \frac{\partial^2 h}{\partial y^2}(\Delta, y) \tag{3.1}$$

Continuity of the $\sigma(y)$ function, and positivity on the interior of the state space suffice for the existence of an infinite-dimensional set of final conditions, such that the solution to (3.1) (with final condition) is not only analytic in a neighborhood of the origin, but entire. This can be seen by a simple construction. Consider an interval $y \in [c, d]$ on which the function $\sigma^2(y)$ is continuous and positive, and let $\sigma_{\min}$ and $\sigma_{\max}$ denote the minimum and maximum values of $\sigma(y)$ on this interval. (Without loss of generality, we take $\sigma(y) > 0$.) We begin with $a_0(y) = 1$ and $a_1(y) = y$, which are solutions to the PDE. Given $a_n(y)$ defined for some integer $n \geq 0$, we define:

$$a_{n+2}(y) = \int_{y_0}^{y} \int_{y_0}^{v} \frac{a_n(u)}{\sigma^2(u)} du dv$$

for some $y_0 \in [c, d]$. Note that the functions:

$$\begin{aligned} h_{0,n}(\Delta, y) &= \sum_{i=0}^{n} \frac{a_{2i}(y)}{(n-i)!} \left(\frac{\Delta}{2}\right)^{(n-i)} \\ h_{1,n}(\Delta, y) &= \sum_{i=0}^{n} \frac{a_{2i+1}(y)}{(n-i)!} \left(\frac{\Delta}{2}\right)^{(n-i)} \end{aligned}$$

are solutions to (3.1) with final conditions $h_{0,n}(0, y) = a_{2n}(y)$ and $h_{1,n}(0, y) = a_{2n+1}(y)$. The $h_{0,n}(\Delta, y)$ and $h_{1,n}(\Delta, y)$ are polynomials in Δ, and therefore everywhere analytic. The finite linear combinations of the $h_{0,n}(\Delta, y)$ and $h_{1,n}(\Delta, y)$:

$$h(\Delta, y) = \sum_{j=0}^{k} c_j h_{0,j}(\Delta, y) + \sum_{j=0}^{k} d_j h_{1,j}(\Delta, y)$$

are also solutions to (3.1) with final condition:

$$h(0, y) = \sum_{j=0}^{k} c_j a_{2j}(y) + \sum_{j=0}^{k} d_j a_{2j+1}(y)$$

Such functions are also everywhere analytic in Δ. The $a_n(y)$ form an infinite-dimensional space of functions, as can be shown by the following argument. Consider only $a_n(y)$ for even n. Then suppose there is some linear combination of a_{2i} for $0 \leq i \leq n$ such that:

$$\sum_{j=0}^{\frac{n}{2}} c_j a_{2j}(y) = 0 \tag{3.2}$$

Then:

$$h(\Delta, y) = \sum_{j=0}^{\frac{n}{2}} c_j h_{0,j}(\Delta, y) \tag{3.3}$$

is a solution to the PDE (3.1) for all Δ, with final condition $h(0, y) = 0$. However, by plugging the power series representation of $h(\Delta, y)$ into the PDE with final condition, it must be the case that $h(\Delta, y) = 0$. If $c_{\frac{n}{2}} \neq 0$, then $c_{\frac{n}{2}} h_{0,\frac{n}{2}}(\Delta, y)$ contains a term of order $\Delta^{\frac{n}{2}}$; none of the other terms on the right-hand side of (3.3) do. Consequently, it must be the case that $c_{\frac{n}{2}} = 0$. In other words, there is no linear combination with $c_{\frac{n}{2}} \neq 0$ that satisfies (3.2), and $a_n(y)$ is linearly independent of the $a_{2i}(y)$ for $0 \leq i < \frac{n}{2}$. By similar argument, the $a_n(y)$ for odd n form an infinite-dimensional set of linearly independent final conditions, which give rise to an infinite-dimensional family of analytic solutions to the PDE.

Infinite linear combinations of the $h_{0,n}(\Delta, y)$ and $h_{1,n}(\Delta, y)$, provided they converge uniformly on all compact subsets in an open neighborhood of Δ and y, are also solutions to (3.1) in that neighborhood. It is possible that an infinite linear combination may converge for some values of Δ but not for others. Consider some d_i, $i \geq 0$, and define:

$$h(\Delta, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{d_{k+l} a_{2l}(y)}{k!} \left(\frac{\Delta}{2}\right)^k + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{d_{k+l} a_{2l+1}(y)}{k!} \left(\frac{\Delta}{2}\right)^k$$

Provided the d_i are chosen such that the sums converge uniformly on some compact set of Δ and y, it can be seen, by term-by-term differentiation, that $h(\Delta, y)$ is a solution (within that compact set) to the PDE (3.1). From the definition of the $a_n(y)$ and the bounds on $\sigma^2(y)$, it follows that:

$$\frac{|y - y_0|^{2n}}{(2n)!\sigma_{\max}^{2n}} \leq |a_{2n}(y)| \leq \frac{|y - y_0|^{2n}}{(2n)!\sigma_{\min}^{2n}}$$

$$\frac{|y - y_0|^{2n+1}}{(2n+1)!\sigma_{\max}^{2n}} \leq |a_{2n+1}(y)| \leq \frac{|y - y_0|^{2n+1}}{(2n+1)!\sigma_{\min}^{2n}}$$

Then:

$$|h(\Delta, y)| \leq \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{|d_{k+l}|}{k!} \frac{|y - y_0|^{2l}}{(2l)!\sigma_{\min}^{2l}} \left(\frac{\Delta}{2}\right)^k + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{|d_{k+l}|}{k!} \frac{|y - y_0|^{2l+1}}{(2l+1)!\sigma_{\min}^{2l}} \left(\frac{\Delta}{2}\right)^k$$

If, for example, the d_i are uniformly bounded, then the above expression converges for all Δ and all $y \in [c, d]$, and is a solution to the PDE (3.1) for such values. However, existence of a solution does not require the d_i to be bounded; if they grow at a sufficiently constrained rate (as a function of i), then they specify a PDE solution for a limited range of Δ.

The PDE (3.1) may appear to be a very restricted special case of the general scalar PDE:

$$\frac{\partial f}{\partial \Delta}(\Delta, x) = \mu(x) \frac{\partial f}{\partial x}(\Delta, x) + \frac{\sigma^2(x)}{2} \frac{\partial^2 f}{\partial x^2}(\Delta, x) - r(x) f(\Delta, x) \tag{3.4}$$

However, this PDE can be converted to (3.1) by changes of variables. Specifically, if we choose:

$$f(\Delta, x) = w(x) h(\Delta, x)$$

where $w(x)$ is a solution to:

$$\mu(x)\, w'(x) + \frac{\sigma^2(x)}{2} w''(x) - r(x)\, w(x) = 0$$

then (3.4) is equivalent to the PDE:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \left[\mu(x) + \sigma^2(x) \frac{w'(x)}{w(x)}\right] \frac{\partial h}{\partial x}(\Delta, x) + \frac{\sigma^2(x)}{2} \frac{\partial^2 h}{\partial y^2}$$

This change of dependent variables thus eliminates the coefficient on $h(\Delta, y)$ from the PDE. A further change of independent variable, using the scale transformation, eliminates the coefficient on the first spatial derivative as well, so the transformed PDE is of the form of (3.1) (but with a different coefficient on the second spatial derivative). Thus, despite its apparently restrictive appearance, the results on analytic solutions to (3.1) carry over to a much more general class of PDE. Given only modest smoothness properties on the coefficients of (3.4), there exists an infinite-dimensional class of final conditions such that an everywhere analytic (in Δ) solution exists.

We have now characterized a set of final conditions for which the solution to (3.1) (and other scalar PDEs that can be transformed to (3.1) by change of variables) is analytic in Δ. However, this characterization may not always be very useful in practice. It is relatively straightforward to construct final conditions that generate analytic solutions, but it is less obvious how to take a given final condition and determine whether it is in fact spanned by the $a_n(y)$ functions specified above. However, in several special cases, there are other techniques for characterizing the set of final conditions that correspond to analytic (in Δ) solutions. The following section explores these cases.

4. Analytic Solutions to Scalar Diffusion Problems

The results of Section 2 allow us to construct power series representations of solutions to conditional moment or asset pricing problems; the results of Section 3 show that, for essentially any scalar diffusion problem, there exists a large, non-trivial class of such problems for which the power series converge. However, in practice, it may be difficult to determine whether, for a given diffusion, the final condition is such that the series does indeed converge. Well-known results from complex analysis establish that power series converge if the solution to the conditional moment or asset pricing problem has the necessary smoothness properties; however, if the solution were known explicitly, there would be no need to find a power series for it. In this section, we consider the problem of determining when, given only the general PDE (that is, the dynamics of the economy) and the final condition (that is, the particular conditional moment or contingent claim price sought), a solution has the analyticity and other properties needed to apply these results. Although our focus in this section is on scalar diffusion and asset pricing problems, multiple diffusion problems can sometimes be decomposed into a system of scalar problems; for example, if two state variables follow independent diffusions, and enter into the interest rate function additively, then the bond pricing problem can be decomposed into a system of two scalar problems. The methods of this section therefore, despite their focus on scalar diffusions, have some applicability to multivariate diffusion problems.

We first describes some changes of variables that can be used to convert scalar conditional moment or pricing problems into a canonical form. The rest of the section explores two particular classes of diffusions in detail, characterizing the region of analyticity of the solution to the conditional moment or pricing problem, thereby allowing us to know when a power series converges, and what the range of convergence is. For these two classes of problems, the region of analyticity of the solution depends critically on certain smoothness and growth conditions. Specifically, lack of smoothness (i. e., non-analyticity of the final condition) or growth at a rate faster than $c\exp\left(kx^2\right)$ for any $c, k > 0$, in any direction in the complex plane, results in a singularity at the origin. Smoothness (i. e., analyticity of the final condition) and growth at a rate bounded by $c\exp\left(kx^2\right)$ for some $c, k > 0$ in all directions in the complex plane results in a region of analyticity around the origin, with the size of this region determined by the k parameter. Smoothness and existence of a bound of the form $c_k\exp\left(kx^2\right)$ for any $k > 0$ results in analyticity (and therefore convergence of power series) for all values of the time variable.

4.1. Canonical Form PDE

Change of independent variable is a technique frequently used to simplify analysis of a diffusion process (or, equivalently, a parabolic partial differential equation). Less used in the finance and economics literature are changes of time variable and dependent variable, although the latter technique has been used in the partial differential equation literature; see, for example, Colton (1979).[9] By the use of such transformations, solution of the general Feynman-Kac problem can often be reduced to solution of a special case, although, many such transforms cannot easily be applied to multivariate diffusions.[10] Focusing on the scalar diffusion case, we consider the problem:

$$\frac{\partial f}{\partial \Delta}(\Delta, x) = \mu(x)\frac{\partial f}{\partial x}(\Delta, x) + \frac{\sigma^2(x)}{2}\frac{\partial^2 f}{\partial x^2}(\Delta, x) - r(x) f(\Delta, x) \tag{4.1}$$

with final condition $f(0, x) = g(x)$. We seek solutions to this equation for all $x \in (a, b)$ where a and b are the boundaries of the diffusion process (with $a = -\infty$ or $b = +\infty$ or both possible) and $\Delta \in [0, T]$ for some $T > 0$. We require $\sigma(x) \neq 0$ for all $x \in (a, b)$; as the PDE is motivated by a diffusion process, it will usually be the case that $\mu(x)$ and $\sigma(x)$ are chosen so that the boundaries a and b cannot be reached in finite time. However, it is quite possible to analyze the PDE problem without imposing such restrictions. In general, there are multiple solutions to the PDE problem, even with the given final condition, but at most one of these solutions is also the solution to the probabilistic problem.[11] However, there can be at most one solution which

[9]The pricing of derivative securities through use of risk-neutral or other artificial probability measures could be viewed as similar, since finding the risk-neutral expectation of a function is equivalent to finding the expectation (under true probabilities) of the same function multiplied by the Radon-Nikodym derivative. Note, however, the difference. Risk-neutral pricing involves the expectation of a random variable multiplied by the Radon-Nikodym derivative, which effectively changes the probability measure. Here, we multiply the expectation itself by a factor that changes the dependent variable.

[10]The changes of variables used in Section 3 are essentially the inverse of the transformations of Colton (1979) used here.

[11]For example, the function that is everywhere zero is a solution to the ordinary heat equation with a final condition of zero, and is also the correct solution to the corresponding probabilistic problem, since the expected value of zero is trivially zero at all

is analytic in the time variable, and, if it exists, this solution is the one that also solves the corresponding probabilistic problem.

Several different changes of variables have been used to simplify stochastic processes and/or partial differential equations. The scale transformation (see, for example, Karlin and Taylor (1981)) is a change of independent variable often used to eliminate the drift from a diffusion process (or, equivalently, to remove the first spatial derivative term from a PDE). Aït-Sahalia (2002) uses a different change of independent variable to normalize the diffusion coefficient of a stochastic differential equation to one (or, equivalently, to set the coefficient of the second spatial derivative in a PDE to one half). Colton (1979) transforms both the dependent and independent variables, allowing both the elimination of the first spatial derivative term from the PDE, and normalization of the second spatial derivative coefficient to one half. We employ this latter technique, using the transforms:

$$y = \int^x \frac{du}{\sigma(u)}$$

$$f(\Delta, x) = \exp\left(-\int^x \left[\frac{\mu(u)}{\sigma^2(u)} - \frac{\sigma'(u)}{2\sigma(u)}\right] du\right) h(\Delta, y)$$

Note that the lower limits of the integrals are not specified, so these expressions really describe a family of transforms. Positivity and continuity of $\sigma(x)$ on the interior of the state space ensure that y is a strictly increasing function of x, and can therefore be inverted.

The transformed differential equation, expressed in terms of y and $h(\Delta, y)$ instead of x and $f(\Delta, x)$, is:

$$\frac{\partial h}{\partial \Delta} = \frac{1}{2}\frac{\partial^2 h}{\partial y^2} - r_h(y)\, h \tag{4.2}$$

with:

$$r_h(y) \equiv -\frac{\mu^2(x)}{2\sigma^2(x)} - \frac{\mu'(x)}{2} + \frac{\mu(x)\,\sigma'(x)}{\sigma(x)} - \frac{\sigma'(x)\,\sigma'(x)}{8} + \frac{\sigma''(x)\,\sigma(x)}{4} - r(x)$$

where x is an implicit function of y. The final condition expressed in terms of h and y is:

$$h(0, y) = g_h(y) \equiv \exp\left(\int^x \left[\frac{\mu(u)}{\sigma^2(u)} - \frac{\sigma'(u)}{2\sigma(u)}\right] du\right) g(x) \tag{4.3}$$

Note that, provided the diffusion coefficient is bounded away from zero on the interior of the state space, these transforms are always well-defined (although in some cases we may not be able to evaluate the integral in the transformed final condition explicitly). Since y as a function of x can be inverted, the process Y_t, defined by applying the change of independent variables to X_t, inherits the Markov property of X_t. On the interior of the state space of X_t, the ratio between f and h is positive, so, for example, a strictly positive f implies a strictly positive h.

We can also assign probabilistic meaning to the transformed PDE given in (4.2); this same equation (given

time horizons. However, there also exist non-zero solutions to the same PDE with the same final condition; see, for example, the construction in Cannon (1984). The alternate solution is necessarily non-analytic at $\Delta = 0$; if it were analytic, the coefficients of the power series would have to satisfy the recursive relation derived in Section 2, and with a final condition equal to zero, this relation can only be satisfied if the coefficients are all zero.

sufficient regularity conditions) arises as the solution to the probabilistic problem:

$$h(\Delta, y) = E\left[g_h(W_{t+\Delta})\exp\left(-\int_t^{t+\Delta} r_h(W_u)du\right)\big|W_t = y\right]$$

where W_t is a canonical Brownian motion. Note, however, that although the independent variable in this equation is y, the process Y_t, defined by the change of independent variables applied to X_t, is in general not a Brownian motion, and may have a state space different than the state space of the Brownian motion (i. e., the entire real line). Nonetheless, the change of variable techniques described so far show that the original pricing problem is equivalent to the problem of finding a functional of a Brownian motion, even when the state variable is not a Brownian motion. The asset pricing problem is then equivalent to the problem of pricing a different asset in a different economy, in which both the interest rate and the final payoff of the alternate asset are functions of the value of a Brownian motion.

In a term structure context, the pricing PDEs for models that may seem quite distinct at first can sometimes be transformed by change of variables to the same general PDE, with only the final condition differing. For example, in the scalar version of the linear-quadratic model of Ahn, Dittmar, and Gallant (2002), the $r_h(y)$ coefficient is a quadratic function of y; the model of Vasicek (1977) transforms to the same PDE after application of the change of variables; in both cases, the final condition for a zero-coupon bond price is $g(x) = 1$, which implies that $g_h(y)$ is exponential quadratic (but with different coefficients in the two models). The pricing PDEs for the models of Cox, Ingersoll, and Ross (1985) and Ahn and Gao (1999) both transform to the case in which $r_h(y)$ contains a term proportional to y^2, a constant term, and a term proportional to $1/y^2$; the two models then differ (for bond pricing purposes) only in the parameter values and the specification of g_h. The pricing PDE for callable corporate bonds in the model of Jarrow, Li, Liu, and Wu (2006) also transforms to this case; these authors use our technique to approximate the callable bond prices. However, there also exist many other models that have not yet appeared in the literature, but that also transform to these cases.

In Section 3, it was shown that, for essentially arbitrary choice of $\mu(x)$, $\sigma(x)$, and $r(x)$ functions, there is an infinite-dimensional family of final conditions $g(x)$ such that the corresponding solution $h(\Delta, y)$ of (4.2), with final condition as in (4.3), is analytic in Δ in some neighborhood of the origin. That section also shows how to construct such a $g(x)$. However, the methods of that section are not particularly useful in solving the reverse problem, that of determining, for a given $g(x)$ function, whether $h(\Delta, y)$ is analytic. Determining whether the solution is analytic around the origin is important, since, if it is not, a power series does not converge anywhere else. Even if the solution is analytic in some neighborhood of $\Delta = 0$, it is important to know the locations of any singularities, since the location of such singularities determines the range of convergence of any power series constructed. There exist at least two forms of (4.2) for which the location of the singularities of the solutions can be characterized explicitly. These two forms encompass many, if not all, of the term structure models commonly used in the literature for which bond prices are known explicitly.[12]

[12]The model of Ahn and Gao (1999) is encompassed by one of the two forms discussed here. However, this model is a rare (and perhaps unique) case in that bond prices are not analytic in maturity at $\Delta = 0$, unless strong (and probably unrealistic) restrictions are imposed on the model parameters. Consequently, power series for bond prices under this model do not converge.

However, these two forms also encompass many other potential models that have not as yet been studied. The next two sections examine these two classes of models; see Kimmel (2008a) and Jarrow, Li, Liu, and Wu (2006) for applications of these results.

4.2. Brownian Motion

The special cases of (4.2) in which the $r_h(y)$ coefficient is either linear or quadratic:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - (ay + d)\, h(\Delta, y) \tag{4.4}$$

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left[\frac{b^2}{2}(y - a)^2 + d\right] h(\Delta, y) \tag{4.5}$$

admit particularly straightforward analysis. A constant or zero r_h function is a special case of both (4.4) and (4.4). It is possible, by parameterizing the r_h function in (4.5) slightly differently, to include (4.4) as a special case as well; however, the solutions of the two equations have sufficiently different properties so as to warrant separate treatment. The region of analyticity for a solution $h(\Delta, y)$ of (4.4) and (4.5) can be characterized in a straightforward manner, as shown in the following two theorems. Throughout not only this section but also the next, we take as given a norm $\|z\|$ (over the reals) on the set of all complex numbers z. Note, that $\|z\|$ need not be a norm over the complex numbers, that is, $\|az\| = |a|\,\|z\|$ for all complex z and real a, but not necessarily for complex a. If $\|z\|$ were restricted to be a norm over the complex numbers, the only admissible norms would be multiples of the modulus function. However, norms over the reals include, for example:

$$\|z\| \equiv \sqrt{\frac{(\operatorname{Re} z)^2}{k_1} + \frac{(\operatorname{Im} z)^2}{k_2}}$$

which, for $k_1 \neq k_2$, is not a norm over the complex numbers. Use of norms that are asymmetric with respect to direction in the complex plane can establish regions of analyticity (in the time variable) for PDE solutions that extend further in some directions in the complex plane than in others. For purposes of establishing the region of convergence of a power series, it may seem that this generality serves no purpose; a power series converges within a circle, so only symmetric norms are particularly useful. However, it is possible to construct power series not in Δ, but in some non-affine function of Δ. In this case, appropriate choice of such a non-affine function can effectively extend the range of convergence for positive Δ if analyticity of the solution can be established in a non-circular region. See Kimmel (2008b) for examples. For this reason, we use a general norm, rather than simply multiples of the modulus function, in all the main results.

The following theorem characterizes the region of analyticity of the solution to (4.4).

Theorem 1 *Let $g(y)$ be analytic for all complex y, and let there exist some $c > 0$ and some norm (over the*

To the best of our knowledge, this model is the only interest rate model with closed-form bond prices that cannot be represented by convergent power series, although it should be noted that bond prices are closed-form under this model only if the confluent hypergeometric function (also known as Kummer's function) is taken to be fundamental. Although power series for bond prices do not usually converge in this model, there exist other classes of security prices with convergent power series representations.

reals) $\|y\|$ *such that* $g(y)$ *satisfies the bound:*

$$|g(y)| \leq ce^{\frac{\|y\|^2}{2}}$$

Let a *and* d *be any complex numbers. Then there exists an analytic function* $h(\Delta, y)$*, defined for all complex* y *and* Δ *such that* $\left\|\sqrt{\Delta}\right\| < 1$*, that satisfies:*

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - (ay + d)\, h(\Delta, y) \tag{4.6}$$

$$h(0, y) = g(y) \tag{4.7}$$

Proof: See appendix.

Although the solution to (4.6) and (4.7) obviously depends on the parameters a and d, the region of existence and analyticity established by the theorem does not. This result can be interpreted as follows: if the theorem establishes that the conditional expectation of some function of a Brownian motion is analytic in a particular region, it also establishes that the discounted expected value of the same function is analytic in the same region, if the instantaneous interest rate is an affine function of the state variable. Note, however, that Section 4.1 establishes that the theorem applies to a much broader class of problems than those in which the state variable process is a Brownian motion. Many non-affine problems are covered by the theorem, by the changes of variables described in that section.

By contrast, the quadratic coefficient b in (4.5) has a strong effect on the region in which the PDE solution can be shown to exist and be analytic in Δ. The following theorem addresses this case.

Theorem 2 *Let* $g(y)$ *be analytic for all complex* y*, and let* a*,* b*, and* d *be arbitrary complex numbers. Let there exist some* $c > 0$ *and some norm (over the reals)* $\|y\|$ *such that* $g(y)$ *satisfies the bound:*

$$\left|e^{-\frac{b}{2}(y-a)^2} g(y)\right| \leq ce^{\frac{\|y\|^2}{2}}$$

Define:

$$\tau(\Delta) \equiv \frac{e^{2b\Delta} - 1}{2b} \qquad b \neq 0$$

$$\tau(\Delta) \equiv \Delta \qquad b = 0$$

Then there exists a function $h(\Delta, y)$*, defined and analytic for all complex* y *and* Δ *such that* $\left\|\sqrt{\tau(\Delta)}\right\| < 1$*, that satisfies:*

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left[\frac{b^2}{2}(y - a)^2 + d\right] h(\Delta, y) \tag{4.8}$$

$$h(0, y) = g(y) \tag{4.9}$$

Proof: See appendix.

In probabilistic terms, this theorem describes a large class of functions of a Brownian motion whose conditional expectations are analytic in the time variable, and characterizes the region of analyticity. However, it also applies to many other situations. For example, a process which is not a Brownian motion, but that can be changed to a Brownian motion by change of independent variable, is also covered by applying Theorem 1 or Theorem 2 after the change of variables. Similarly, this theorem effectively characterizes a set of final asset payoffs that generate pricing functions that are analytic in maturity, provided the pricing PDE can be converted to (4.4) or (4.5) by change of dependent and/or independent variables, as described in Section 4.1. For any of these applications, if the conditions of the theorem hold for a symmetric norm of the form $\|z\| \equiv |z| / \sqrt{k_0}$, then the solution to the PDE is analytic for all $|\Delta| < k_0$, and a power series approximation to the solution converges for at least these values of Δ. If the conditions of the theorem hold for an asymmetric norm not of this form, then time transformation methods (see Kimmel (2008b)) may improve the range of convergence.

It may be useful to characterize those final conditions that correspond to solutions $h(\Delta, y)$ of (4.4) and (4.5) that are defined and analytic for all values of Δ. The following two corollaries examine these cases:

Corollary 1 *Let $g(y)$ be analytic for all complex y, and for each positive real $k > 0$, let there exist some $c_k > 0$ such that $g(y)$ satisfies the bound:*

$$|g(y)| \leq c_k e^{\frac{|y|^2}{2k}}$$

Let a and d be any complex numbers. Then there exists an analytic function $h(\Delta, y)$, defined for all complex y and Δ, that satisfies:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2} \frac{\partial^2 h}{\partial y^2}(\Delta, y) - (ay + d) h(\Delta, y)$$
$$h(0, y) = g(y)$$

Proof: See appendix.

Corollary 1 extends the result of Theorem 1; given stronger growth restrictions on $g(y)$, the region of analyticity can be extended to all values of Δ. The next result does the same thing for Theorem 2.

Corollary 2 *Let $g(y)$ be analytic for all complex y, and let a, b, and d be arbitrary complex numbers. For each positive real $k > 0$, let there exist some $c_k > 0$ such that $g(y)$ satisfies the bound:*

$$\left| e^{-\frac{b}{2}(y-a)^2} g(y) \right| \leq c_k e^{\frac{|y|^2}{2k}}$$

Then there exists an analytic function $h(\Delta, y)$, defined for all complex y and Δ, that satisfies:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2} \frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left[\frac{b^2}{2}(y-a)^2 + d \right] h(\Delta, y)$$
$$h(0, y) = g(y)$$

Proof: See appendix.

These corollaries applies to all the same situations described in the discussion of Theorems 1 and 2, provided a stronger growth restrictions on the final condition are imposed. But, if the conditions of either Corollary 1 or 2 are satisfied, then the conditional moment or pricing function is analytic for all complex values of the time variable. A power series approximation to the desired function then converges uniformly on the interval $[0, T]$ for any value $0 < T < +\infty$, although in general, this convergence is not uniform on $[0, +\infty)$.

Several term structure models that have appeared in the literature are covered by Theorems 1 and 2 and by Corollaries 1 and 2 (as are many models that have not previously appeared in the literature). The model of Vasicek (1977) is covered by Corollary 2, and the model of Ahn, Dittmar, and Gallant (2002) is covered by Theorem 2. Power series for zero-coupon bond prices in the former model therefore converge for all maturities; for the latter model, the series converge for some finite range, and diverge for longer maturities. Even for the Vasicek (1977) model, though, the convergence is not uniform, with the result that, for very long maturities, a large number of terms in the power series may be needed before the truncated series is a good approximation to the bond price. However, Kimmel (2008b) shows how to apply a non-affine transformation of the time variable to the bond pricing problem in both of these models, and converts them to a problem covered by Corollary 2. Power series for bond prices under both models then converge for all maturities. He goes on to show, under mild assumptions, that the convergence is uniform in maturity. The value of the results in this section, however, is not the ability to approximate bond prices in models for which those prices are already known in closed-form; rather, it is to approximate price of bonds and other contingent claims, and also conditional moments, in the large class of problems that, by the changes of variables described in Section 4.1, are covered by the two theorems and corollaries. See Section 5 for a discussion of the models of Vasicek (1977) and Ahn, Dittmar, and Gallant (2002), and additional examples.

4.3. General Affine

The following partial differential equation is similar to the two cases considered in the previous section, but contains one extra term not included there:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left(\frac{a}{y^2} + \frac{b^2}{2}y^2 + d\right) h(\Delta, y) \tag{4.10}$$

If $a = 0$, then this PDE is a special case of (4.4) (for $b \neq 0$) or (4.5) (for $b = 0$). We refer to this equation as the general affine PDE; although there may at first or even second glance seem to be nothing particularly affine about (4.10), the problem of finding conditional moments for any scalar affine diffusion can be reduced to solving this equation by changes of variables. The pricing PDE for every scalar affine yield model can also be transformed to this PDE by change of variables, as can the pricing PDE for some non-affine yield models (e. g., Ahn and Gao (1999)). Jarrow, Li, Liu, and Wu (2006) consider a model in which the problem of pricing callable corporate bonds reduces, after change of variables, to the problem of solving this PDE with a particular final condition. The following theorem characterizes explicitly the final conditions that admit analytic (in Δ) solutions.

Theorem 3 *Let $g_1(y)$ and $g_2(y)$ be even and analytic for all complex y, and let a, b, and d be constants. Let there exist some $c > 0$ and some norm (over the reals) $\|y\|$ such that $g_1(y)$ and $g_2(y)$ satisfy the bounds:*

$$\left|e^{-\frac{b}{2}y^2} g_1(y)\right| \leq c e^{\frac{\|y\|^2}{2}} \qquad \left|e^{-\frac{b}{2}y^2} g_2(y)\right| \leq c e^{\frac{\|y\|^2}{2}} \tag{4.11}$$

Define:

$$\tau(\Delta) \equiv \frac{e^{2b\Delta}-1}{2b} \qquad b \neq 0$$
$$\tau(\Delta) \equiv \Delta \qquad b = 0$$

Then there exist analytic functions $h_1(\Delta, y)$ *and* $h_2(\Delta, y)$*, defined for all complex* y *and* $\left\|\sqrt{\tau(\Delta)}\right\| < 1$*, such that* $h(\Delta, y)$*, defined by:*

$$h(\Delta, y) \equiv h_1(\Delta, y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + h_2(\Delta, y)\, y^{\frac{1+\sqrt{1+8a}}{2}} \qquad \frac{\sqrt{1+8a}}{2} \notin \mathbb{N}$$
$$h(\Delta, y) \equiv h_1(\Delta, y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + h_2(\Delta, y)\, y^{\frac{1+\sqrt{1+8a}}{2}} \ln y \qquad \frac{\sqrt{1+8a}}{2} \in \mathbb{N}$$

satisfies the partial differential equation:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left(\frac{a}{y^2} + \frac{b^2}{2}y^2 + d\right) h(\Delta, y) \tag{4.12}$$

with final condition:

$$h(0, y) = g_1(y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + g_2(y)\, y^{\frac{1+\sqrt{1+8a}}{2}} \qquad \frac{\sqrt{1+8a}}{2} \notin \mathbb{N} \tag{4.13}$$
$$h(0, y) = g_1(y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + g_2(y)\, y^{\frac{1+\sqrt{1+8a}}{2}} \ln y \qquad \frac{\sqrt{1+8a}}{2} \in \mathbb{N} \tag{4.14}$$

for all complex $y \neq 0$ *and* $\left\|\sqrt{\tau(\Delta)}\right\| < 1$*. Throughout,* $\sqrt{1+8a}$ *refers to the positive square root when* a *is real and* $a \geq -1/8$*.*

Proof: See appendix.

If $a = 0$, this PDE reduces to that of either Theorem 1 or Theorem 2, and $h(\Delta, y)$ is a solution to the PDE at $y = 0$ as well. If $a \neq 0$ and $\sqrt{1+8a}$ is an odd integer, then $h(\Delta, y)$ generally has a pole of order $\left(\sqrt{1+8a}-1\right)/2$ at $y = 0$, but is analytic at this point in the special case $g_1(y) = 0$. If $\sqrt{1+8a}$ is not an odd integer, then the solution is never analytic at $y = 0$ (except for the trivial special case of $g_1(y) = g_2(y) = 0$), and there is also a branch cut discontinuity in the complex plane. In this case, we take $h(\Delta, y)$ to be a so-called *global analytic function*, which is an equivalence class of overlapping branches of the analytic continuation of a function (see, for example, Ahlfors (1979)). The solution can then be considered to be analytic for all values of $y \neq 0$.

The above result, while valid for any value of a, may not be expressed in the most useful form for $a < -1/8$, since the two terms in the expression for $h(0, y)$ are then, in general, complex, even if $g_1(y)$ and $g_2(y)$ are real functions; for most applications, we are interested in real-valued final conditions. However, the final condition can be restated equivalently as:

$$h(0, y) = g_3(y)\sqrt{y}\cos\left(\frac{\sqrt{-8a-1}}{2}\right) + g_4(y)\sqrt{y}\sin\left(\frac{\sqrt{-8a-1}}{2}\right)$$

where $g_3(y) = g_1(y) + g_2(y)$ and $g_4(y) = \imath [g_2(y) - g_1(y)]$ or, equivalently, $g_1(y) = [g_3(y) + \imath g_4(y)]/2$ and $g_2(y) = [g_3(y) - \imath g_4(y)]/2$. If the functions $g_1(y)$ and $g_2(y)$ satisfy the growth conditions of (4.11), then $g_3(y)$ and $g_4(y)$ satisfy the same growth conditions (possibly with a different value of c); the final condition $g(y)$ is then real, provided $a < -1/8$ and $g_3(y)$ and $g_4(y)$ are real functions.

As in the Brownian motion case, we may interpret solutions to (4.10) as functionals of a Brownian motion; specifically, they are the expected value of the final condition, applied to the terminal value of the Brownian motion, discounted at an interest rate specified by the last term in the PDE:[13]

$$h(\Delta, y) = E\left[g(W_{t+\Delta}) \exp\left(-\int_t^{t+\Delta} \left(\frac{a}{W_u^2} + \frac{b^2}{2} W_u^2 + d \right) du \right) \middle| W_t = y \right]$$

The theorem describes a large class of final conditions such that the corresponding solutions $h(\Delta, y)$ are analytic in the time variable, and characterizes the region of analyticity. As in the Brownian motion case, it also applies to many other situations. For example, conditional moments of the square-root process of Feller (1951) (after change of independent variable) satisfy this PDE. Furthermore, conditional moments of any process that can be changed to the square-root process by a change of independent variable are also covered the theorem. Similarly, the theorem effectively characterizes a set of final asset payoffs that generate pricing functions which are analytic in maturity, for a wide combination of process and interest rate specifications, provided the pricing PDE can be converted to (4.10) by change of dependent and/or independent variables, as in Colton (1979).

As in the Brownian motion case, it may be useful to characterize those final conditions that correspond to solutions $h(\Delta, y)$ to (4.10) that are analytic for all values of Δ. The following corollary examines this case:

Corollary 3 *Let $g_1(y)$ and $g_2(y)$ be even and analytic for all complex y, and for each positive real k, let there exist some $c_k > 0$ such that $g_1(y)$ and $g_2(y)$ satisfy the bounds:*

$$|g_1(y)| \leq c_k e^{\frac{|y|^2}{2k}} \qquad |g_2(y)| \leq c_k e^{\frac{|y|^2}{2k}}$$

Then for any complex a, there exist analytic functions $h_1(\Delta, y)$ and $h_2(\Delta, y)$, defined for all complex y and Δ, such that $h(\Delta, y)$, defined as:

$$h(\Delta, y) \equiv h_1(\Delta, y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + h_2(\Delta, y)\, y^{\frac{1+\sqrt{1+8a}}{2}} \qquad \frac{\sqrt{1+8a}}{2} \notin \mathbb{N}$$

$$h(\Delta, y) \equiv h_1(\Delta, y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + h_2(\Delta, y)\, y^{\frac{1+\sqrt{1+8a}}{2}} \ln y \qquad \frac{\sqrt{1+8a}}{2} \in \mathbb{N}$$

satisfies the partial differential equation:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2} \frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left(\frac{a}{y^2} + \frac{b^2}{2} y^2 + d \right) h(\Delta, y)$$

[13] This interpretation is subject to technical conditions that make the probabilistic problem equivalent to the partial differential equation. However, as none of the subsequent analysis depends on the applicability of this interpretation, we do not verify these conditions.

with final condition:

$$h(0,y) = g_1(y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + g_2(y)\, y^{\frac{1+\sqrt{1+8a}}{2}} \qquad \frac{\sqrt{1+8a}}{2} \notin \mathbb{N}$$

$$h(0,y) = g_1(y)\, y^{\frac{1-\sqrt{1+8a}}{2}} + g_2(y)\, y^{\frac{1+\sqrt{1+8a}}{2}} \ln y \qquad \frac{\sqrt{1+8a}}{2} \in \mathbb{N}$$

for all complex $y \neq 0$ and Δ.

Proof: See appendix.

This corollary applies to all the same situations described in the discussion of Theorem 3, provided the stronger growth restriction on the final condition is imposed. But, if the conditions of the corollary apply, then the conditional moment or pricing function is analytic for all complex values of the time variable, and a power series to the desired function converges for all values of Δ. However, in general, this convergence is not uniform on $\Delta \in [0, +\infty)$.

As in the Brownian motion case, these previous two results can be applied to a broad class of problems. As previously noted (and shown in detail in Section 5), several term structure models that have appeared in the literature reduce to the Brownian motion case after changes of variables; several more reduce to the general affine case (as do many other models that have not previously appeared in the literature). Furthermore, better convergence properties can often be established by change of the time variable before application of Theorem 3 or Corollary 3 this method often extends the range of Δ for which a power series converges, and sometimes even establishes uniform convergence for all positive Δ. See Kimmel (2008b).

5. Examples

In this section, we approximate solutions, by truncated power series, to the conditional moment and bond pricing problem in several models. Some of these models have appeared in the literature, and the solutions are known in closed-form; in these cases, the approximate solutions can be compared to the exact solutions to evaluate the accuracy of the approximation. Other cases we examine have not previously appeared in the literature, and exact solutions are unknown. All the cases we consider are covered by the results of Section 4, which demonstrate that power series for these solutions converge, and which also establish the range of convergence.

5.1. Vasicek Term Structure Model

In the well-known term structure model of Vasicek (1977), the risk-neutral interest rate process is of the form:

$$dr_t = \kappa(\theta - r_t)\, dt + \sigma dW_t$$

Consider the time t price $P(\Delta, r)$ of a zero-coupon bond that pays one unit of account at time $t+\Delta$, with the current short interest rate given by $r_t = r$. This price satisfies the PDE with final condition:

$$\frac{\partial P}{\partial \Delta}(\Delta, r) = \kappa(\theta - r)\frac{\partial P}{\partial r}(\Delta, r) + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial r^2}(\Delta, r) - rP(\Delta, r)$$
$$P(0, r) = 1$$

We first use the changes of variables of Section 4.1 to express the PDE in the canonical form:

$$P(\Delta, r) = e^{\frac{\kappa}{2}\left[y(r) - \frac{\theta}{\sigma}\right]^2} h(\Delta, y(r))$$
$$y(r) = \frac{r}{\sigma}$$

By substitution of this expression for $P(\Delta, r)$ into the original pricing PDE, we find that $h(\Delta, y)$ must satisfy the canonical form PDE with final condition:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left[\frac{\kappa^2}{2}\left(y - \frac{\theta}{\sigma}\right)^2 + \sigma y - \frac{\kappa}{2}\right] h(\Delta, y)$$
$$h(0, y) = e^{-\frac{\kappa}{2}\left(y - \frac{\theta}{\sigma}\right)^2}$$

This problem is covered by Corollary 2, which establishes existence of a solution that is everywhere analytic in Δ, with:

$$b = -\kappa$$
$$a = \frac{\theta}{\sigma} - \frac{\sigma}{\kappa^2}$$
$$d = \theta - \frac{\kappa}{2} - \frac{\sigma^2}{2\kappa^2}$$

We note that:

$$e^{-\frac{b}{2}(y-a)^2} g(y) = e^{\frac{\kappa}{2}\left(y - \frac{\theta}{\sigma} + \frac{\sigma}{\kappa^2}\right)^2} e^{-\frac{\kappa}{2}\left(y - \frac{\theta}{\sigma}\right)^2} = e^{y\frac{\sigma}{\kappa} - \frac{\theta}{\kappa} + \frac{\sigma^2}{2\kappa^3}}$$

This is the quantity to which the bound of Corollary 2 applies; since, for any $k > 0$, there exists a $c_k > 0$ such that:

$$\left| e^{-\frac{b}{2}(y-a)^2} g(y) \right| = \left| e^{y\frac{\sigma}{\kappa} - \frac{\theta}{\kappa} + \frac{\sigma^2}{2\kappa^3}} \right| \leq c_k e^{\frac{|y|^2}{2k}}$$

the bound is satisfied. Note that the bound would not be satisfied if we had chosen $b = \kappa$ instead of $b = -\kappa$, even though Theorem 2 allows either choice. If we had chosen $b = \kappa$, then the conditions of Theorem 2 would be satisfied for some norms and not others, so that the theorem would establish a region of analyticity around $\Delta = 0$. However, Corollary 2 would not apply, so this choice would not establish analyticity everywhere of the PDE solution. The choice of $b = -\kappa$ therefore establishes a stronger result.

Application of Corollary 2 (with $b = -\kappa$) therefore establishes that $h(\Delta, y)$ is everywhere analytic in Δ and in y. From the relation between $P(\Delta, r)$ and $h(\Delta, y)$, it follows that $P(\Delta, r)$ is also analytic in both Δ

and r. We can therefore construct a power series representation for either $h(\Delta, y)$ or $P(\Delta, r)$, calculating the coefficients using (2.6) and (2.7); in either case, the power series converges for all Δ, irrespective of the value of y or r. The first few coefficients in the power series for $h(\Delta, y)$ are:

$$a_0(y) = e^{-\frac{\kappa}{2}\left(y-\frac{\theta}{\sigma}\right)^2}$$
$$a_1(y) = e^{-\frac{\kappa}{2}\left(y-\frac{\theta}{\sigma}\right)^2}(-y\sigma)$$
$$a_2(y) = e^{-\frac{\kappa}{2}\left(y-\frac{\theta}{\sigma}\right)^2}\left(y^2\sigma^2 + y\kappa\sigma - \theta\kappa\right)$$
$$a_3(y) = e^{-\frac{\kappa}{2}\left(y-\frac{\theta}{\sigma}\right)^2}\left[-y^3\sigma^3 - 3y^2\kappa\sigma^2 + \left(3\theta\kappa - \kappa^2\right)y\sigma + \theta\kappa^2 + \sigma^2\right]$$

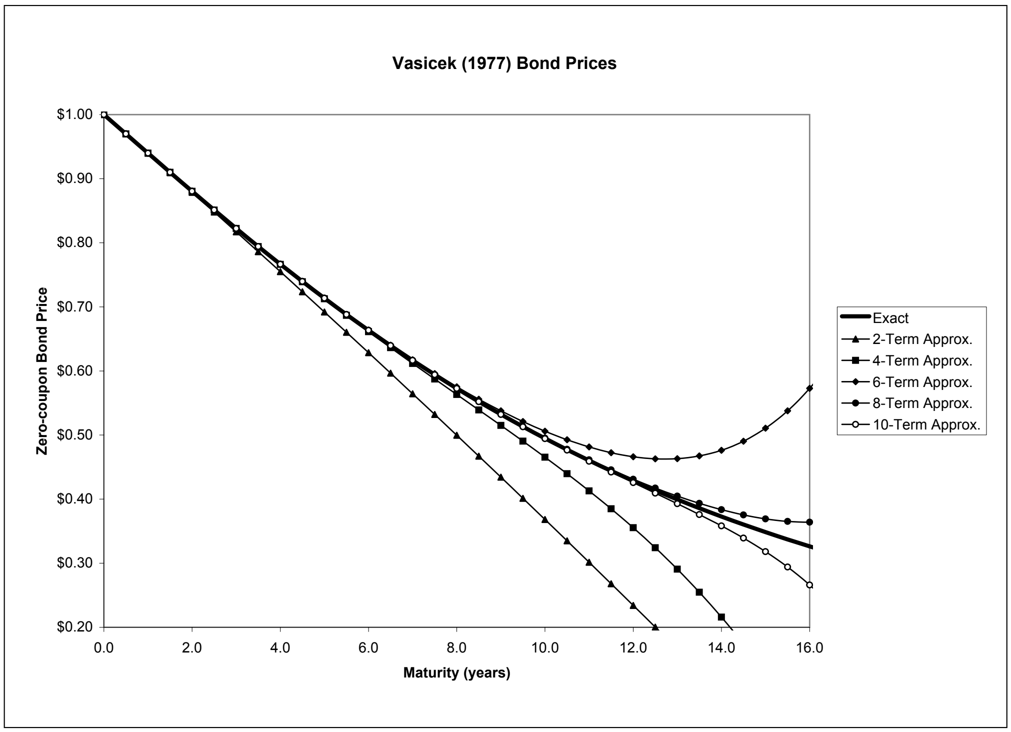

Figure 1: This figure shows prices of zero-coupon bonds as a function of maturity, from the model of Vasicek (1977), both in closed-form and with approximations including varying numbers of terms. The risk-neutral parameters are from Cheridito, Filipović, and Kimmel (2007), and the current instantaneous interest rate is taken to be $r = 6\%$. As shown, the two-term approximate price quickly deviates substantially from the true price as maturity increases beyond a few years. However, approximations with more terms do much better; the error in yield of the 10-term approximation for a bond with maturity of ten years is approximately one basis point. For this model, bond price approximations converge for all maturities; however, the convergence is not uniform, and might therefore be slow for very large maturities.

Figure 1 compares the true prices of zero-coupon bonds under the model of Vasicek (1977), to prices obtained using the approximations from truncated power series as described here. The risk-neutral parameter values used are from Cheridito, Filipović, and Kimmel (2007). As shown, the approximations are quite accurate even with a small number of terms for short maturity bonds. As maturity increases, more terms are needed to achieve a given level of accuracy; ten term approximations are very accurate for maturities up to at least ten years. The need for more terms is a consequence of the type of convergence of the series; although they converge for all maturities, they do not converge uniformly. Consequently, a large number of terms may be needed for accuracy when the maturity is very long. By combining our methods with the time transformation techniques of Kimmel (2008b), it is possible to improve dramatically the convergence properties; he considers this particular model, and finds a combination of the two methods results in approximations that are extremely accurate for all maturities, even with only a few terms.

5.2. Ahn, Dittmar, and Gallant

Another model in which bond prices are known in closed-form is that of Ahn, Dittmar, and Gallant (2002), so this model also serves as an illustrative example. The risk-neutral state variable process is the same as in the Vasicek (1977) model, but in this case, the state variable cannot be identified with the instantaneous interest rate:

$$dx_t = \kappa\left(\theta - x_t\right)dt + \sigma dW_t$$

The interest rate process is specified by:[14]

$$r_t = x_t^2 + \phi$$

The pricing PDE, with final condition, for a zero-coupon bond in this model is:

$$\begin{aligned} \frac{\partial P}{\partial \Delta}\left(\Delta, x\right) &= \kappa\left(\theta - x\right)\frac{\partial P}{\partial x}\left(\Delta, x\right) + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}\left(\Delta, x\right) - \left(x^2 + \phi\right)P\left(\Delta, x\right) \\ P\left(0, x\right) &= 1 \end{aligned}$$

The change of variables needed to put the PDE in the canonical form are exactly the same changes as in the Vasicek (1977) model, but with x in place of r. The canonical form PDE is then:

$$\begin{aligned} \frac{\partial h}{\partial \Delta}\left(\Delta, y\right) &= \frac{1}{2}\frac{\partial^2 h}{\partial y^2}\left(\Delta, y\right) - \left[\frac{\kappa^2}{2}\left(y - \frac{\theta}{\sigma}\right)^2 + \sigma^2 y^2 + \phi - \frac{\kappa}{2}\right]h\left(\Delta, y\right) \\ h\left(0, y\right) &= e^{-\frac{\kappa}{2}\left(y - \frac{\theta}{\sigma}\right)^2} \end{aligned}$$

[14]Our parameterization is different and slightly less general than that of Ahn, Dittmar, and Gallant (2002), but suffices for our purposes.

This canonical form PDE is similar to that derived in the Vasicek (1977) case, in that the last term is of the same functional form, but with different parameter values:

$$\begin{aligned}
b &= -\sqrt{\kappa^2 + 2\sigma^2} \\
a &= \frac{\kappa^2\theta}{\sigma\left(\kappa^2 + 2\sigma^2\right)} \\
d &= -\frac{\kappa}{2} + \frac{\theta^2\kappa^2}{\kappa^2 + 2\sigma^2} + \phi
\end{aligned}$$

This PDE and final condition do not, however, satisfy Corollary 2; instead, we must apply Theorem 2, which establishes converge for a limited range of maturities. In particular, we note that:

$$e^{-\frac{b}{2}(y-a)^2} g(y) = e^{\frac{\sqrt{\kappa^2+2\sigma^2}}{2}\left(y - \frac{\kappa^2\theta}{\sigma(\kappa^2+2\sigma^2)}\right)^2} e^{-\frac{\kappa}{2}\left(y-\frac{\theta}{\sigma}\right)^2}$$

Since there is a non-zero coefficient on y^2 in the exponent (whenever $\sigma > 0$), the conditions of Corollary 2 cannot be satisfied here, as they were in the Vasicek (1977) model. However, this quantity satisfies the boundedness condition of Theorem 2 for the norm $\|y\| \equiv \left|y\sqrt{\sqrt{\kappa^2 + 2\sigma^2} - \kappa}\right|$, so a power series approximation to the PDE solution converges for all:

$$|\Delta| < \left|\frac{1}{\sqrt{\kappa^2 + 2\sigma^2} - \kappa}\right|$$

In the case of Vasicek (1977), our results guarantee convergence of a power series approximation to the PDE solution for all Δ. Here, convergence is guaranteed only for some finite range of values, whose size depends on the κ and σ parameters. Note that, in the typical case of $\kappa > 0$ and $\sigma > 0$, the choice of $b = -\sqrt{\kappa^2 + 2\sigma^2}$ establishes a larger radius of convergence than $b = +\sqrt{\kappa^2 + 2\sigma^2}$, although the conditions of the theorem allow either choice.

Theorem 2 (with $b = -\sqrt{\kappa^2 + 2\sigma^2}$) establishes that $h(\Delta, y)$ is analytic for all y, and for Δ within a circle around the origin. If follows immediately that $P(\Delta, r)$ is also analytic for all r, and for the same region of Δ. Power series coefficients for either $h(\Delta, y)$ or $P(\Delta, r)$ can therefore be found using (2.6) and (2.7). The first few coefficients in the power series for $h(\Delta, y)$ are:

$$\begin{aligned}
a_0(y) &= e^{-\frac{\kappa}{2}\left(y-\frac{\theta}{\sigma}\right)^2} \\
a_1(y) &= e^{-\frac{\kappa}{2}\left(y-\frac{\theta}{\sigma}\right)^2}\left(-y^2\sigma^2 - \phi\right) \\
a_2(y) &= e^{-\frac{\kappa}{2}\left(y-\frac{\theta}{\sigma}\right)^2}\left(y^4\sigma^4 + 2y^2\sigma^2(\kappa + \phi) - 2y\theta\kappa\sigma + \phi^2 - \sigma^2\right) \\
a_3(y) &= e^{-\frac{\kappa}{2}\left(y-\frac{\theta}{\sigma}\right)^2}\left[\begin{array}{l} -y^6\sigma^6 - 3\sigma^4(2\kappa + \phi)y^4 + 6\theta\kappa\sigma^3y^3 - \sigma^2\left(4\kappa^2 - 7\sigma^2 + 6\kappa\phi + 3\phi^2\right)y^2 \\ +6\theta\kappa\sigma(\kappa + \phi)y - 2\theta^2\kappa^2 + 2\kappa\sigma^2 + 3\sigma^2\phi - \phi^3 \end{array}\right]
\end{aligned}$$

Figure 2 compares the true prices of zero-coupon bonds under the model of Ahn, Dittmar, and Gallant (2002). These authors do not estimate a single-factor model, so we use parameters that match the unconditional mean, variance, and kurtosis of the interest rate process implied by the CIR model estimated by Cheridito, Filipović, and Kimmel (2007). These three constraints do not identify the four parameters of the

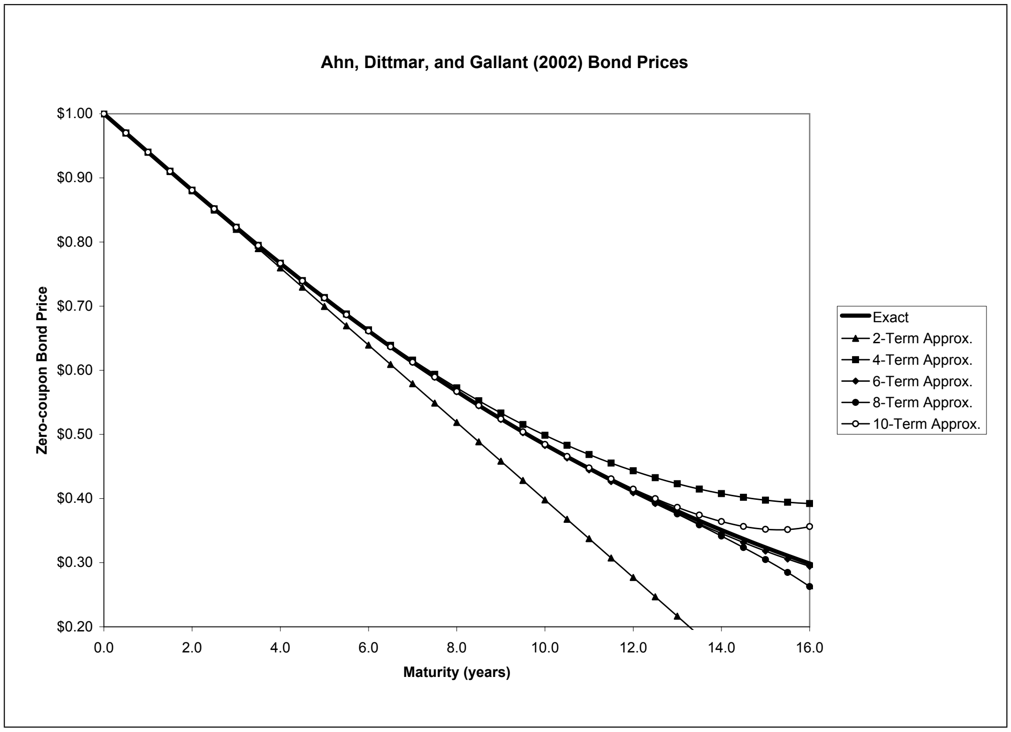

Figure 2: This figure shows prices of zero-coupon bonds as a function of maturity, from the model of Vasicek (1977), both in closed-form and with approximations including varying numbers of terms. The risk-neutral parameters are from Cheridito, Filipović, and Kimmel (2007), and the current instantaneous interest rate is taken to be $r = 6\%$. As shown, the two-term approximate price quickly deviates substantially from the true price as maturity increases beyond a few years. However, approximations with more terms do much better; the error in yield of the 10-term approximation for a bond with maturity of ten years is approximately one basis point. For this model, bond price approximations converge for all maturity; however, the convergence is not uniform, and might therefore be slow for very large maturities.

model, so we also require that the speed of mean reversion be the same as in the CIR model estimates. As shown, the approximations are highly accurate with even a small number of terms, provided the maturity of the bond is short. As maturity increases, however, more terms are needed to achieve a given level of accuracy. Unlike bond prices in the model of Vasicek (1977), bond prices in this model are not everywhere analytic in maturity, and consequently, the approximations converge only for a finite range of maturities. For the particular parameter values chosen, the bond price function has singularities in Δ with modulus of approximately $|\Delta| = 20.8$, which is not coincidentally the range of maturity established by Theorem 2. The power series therefore diverges for longer maturities. Although beyond the scope of this paper, it is possible to improve dramatically the convergence properties of approximations by non-affine transformation of the time variable. See Kimmel (2008b) for examples; he finds that series approximations converge uniformly for bond prices of all maturities when the state variable process is stationary, and even sometimes when it is not. He further finds, in those cases, that the approximations are extremely accurate for all maturities, even with only a few terms.

5.3. Cox, Ingersoll, and Ross

Bond prices are known in closed-form in the model of Cox, Ingersoll, and Ross (1985), and we use this case as an example also. The risk-neutral interest rate process has the same drift as in the Vasicek (1977) model, but a different diffusion term:

$$dr_t = \kappa\left(\theta - r_t\right)dt + \sigma\sqrt{r_t}dW_t$$

See Feller (1951) for restrictions on the parameters that ensure existence of the process, and also non-attainment of the boundary value of zero. The pricing PDE, with final condition, for a zero-coupon bond in this model is:

$$\begin{aligned}\frac{\partial P}{\partial \Delta}\left(\Delta, r\right) &= \kappa\left(\theta - r\right)\frac{\partial P}{\partial r}\left(\Delta, r\right) + \frac{\sigma^2 r}{2}\frac{\partial^2 P}{\partial r^2}\left(\Delta, r\right) - rP\left(\Delta, r\right) \\ P\left(0, r\right) &= 1\end{aligned}$$

The change of variables needed to put the PDE in the canonical form are different than those used in the previous two cases:

$$\begin{aligned}P\left(\Delta, r\right) &= r^{\frac{1}{4}-\frac{\theta\kappa}{\sigma^2}} e^{\frac{\kappa r}{\sigma^2}} h\left(\Delta, y\left(r\right)\right) \\ y\left(r\right) &= \frac{2\sqrt{r}}{\sigma}\end{aligned}$$

The canonical form PDE is then:

$$\begin{aligned}\frac{\partial h}{\partial \Delta}\left(\Delta, y\right) &= \frac{1}{2}\frac{\partial^2 h}{\partial y^2}\left(\Delta, y\right) - \left[\frac{\kappa^2 + 2\sigma^2}{8}y^2 - \frac{\theta\kappa^2}{\sigma^2} + \frac{\left(4\theta\kappa - \sigma^2\right)\left(4\theta\kappa - 3\sigma^2\right)}{8\sigma^4 y^2}\right] h\left(\Delta, y\right) \\ h\left(0, y\right) &= \left(\frac{\sigma y}{2}\right)^{-\frac{1}{2}+\frac{2\theta\kappa}{\sigma^2}} e^{-\frac{\kappa y^2}{4}}\end{aligned}$$

This PDE is of the form specified by Theorem 3, with:

$$\begin{aligned}a &= \frac{\left(4\theta\kappa - \sigma^2\right)\left(4\theta\kappa - 3\sigma^2\right)}{8\sigma^4} \\ b &= -\frac{\sqrt{\kappa^2 + 2\sigma^2}}{2} \\ d &= -\frac{\theta\kappa^2}{\sigma^2}\end{aligned}$$

The boundedness condition of Theorem 3 is imposed on:

$$e^{-\frac{b}{2}y^2} g_2\left(y\right) = e^{\frac{-\kappa + \sqrt{\kappa^2 + 2\sigma^2}}{4}y^2}$$

The theorem conditions are therefore satisfied for $\|y\| \equiv \left|y\sqrt{\sqrt{\kappa^2 + 2\sigma^2} - \kappa}\right|$. A power series approximation to the PDE solution therefore converges for all:

$$|\Delta| < \left|\frac{1}{\sqrt{\kappa^2 + 2\sigma^2} - \kappa}\right|$$

As with the case of Ahn, Dittmar, and Gallant (2002), our results guarantee convergence of a power series approximation to this PDE solution only for a finite range of values of Δ, whose size depends on the κ and σ parameters. Also as in the previous case, it is possible to apply the theorem with the value of b being the negative of the choice above; however, for typical parameter values, the results are stronger with the choice above.

The function $h(\Delta, y)$ is everywhere analytic for all y, and for all Δ within a circle. Analyticity of $P(\Delta, r)$ follows immediately from its relation to $h(\Delta, y)$, so we can therefore construct a power series representation of either function, calculating the coefficients using (2.6) and (2.7). The first few coefficients in the power series of $h(\Delta, y)$ are:

$$
\begin{aligned}
a_0(y) &= \left(\frac{y\sigma}{2}\right)^{-\frac{1}{2}+\frac{2\theta\kappa}{\sigma^2}} e^{-\frac{\kappa y^2}{4}} \\
a_1(y) &= \left(\frac{y\sigma}{2}\right)^{-\frac{1}{2}+\frac{2\theta\kappa}{\sigma^2}} e^{-\frac{\kappa y^2}{4}} \left(-\frac{y^2\sigma^2}{4}\right) \\
a_2(y) &= \left(\frac{y\sigma}{2}\right)^{-\frac{1}{2}+\frac{2\theta\kappa}{\sigma^2}} e^{-\frac{\kappa y^2}{4}} \frac{1}{16}\left(y^4\sigma^4 + 4\kappa\sigma^2 y^2 - 16\kappa\theta\right) \\
a_3(y) &= \left(\frac{y\sigma}{2}\right)^{-\frac{1}{2}+\frac{2\theta\kappa}{\sigma^2}} e^{-\frac{\kappa y^2}{4}} \left[-\frac{\sigma^6 y^6}{64} - \frac{9\kappa\sigma^4 y^4}{48} + \frac{\sigma^2}{4}\left(3\theta\kappa - \kappa^2 + \sigma^2\right) y^2 + \theta\kappa^2\right]
\end{aligned}
$$

Figure 3 compares the true prices of zero-coupon bonds under the model of Cox, Ingersoll, and Ross (1985) to prices obtained using the approximations derived here, using the risk-neutral parameters from Cheridito, Filipović, and Kimmel (2007). As shown, the approximations are very accurate for maturities of eight years or more even with a small number of terms. However, the approximations do not converge, with these parameter values, for maturities greater than approximately 9.17 years. See Kimmel (2008b) for the combination of our technique with time transforms, which extends the range of convergence to arbitrarily long maturities.

5.4. Other Models

In all of the examples considered so far, the quantity sought (the price of a zero-coupon bond) is already known in closed-form; the examples therefore serve as a benchmark to evaluate our technique. The value of an approximation method, however, is not to approximate things that are already known, but things that are unknown. The tools developed here allow one to construct a wide variety of models in which prices or conditional moments are not known in closed-form, but which can nonetheless be approximated accurately. All that is needed is to choose one of the two versions of the canonical PDE examined in Section 4, and specify a final condition that satisfies the conditions of Theorems 1, 2, or 3, or Corollaries 1, 2, or 3. Solutions to these canonical form problems can then be found by series approximation; furthermore, by reversing the changes of dependent and independent variables[15] used to construct the canonical form PDE, a wide variety of non-canonical problems can be constructed, all of which can be solved by series approximation.

Continuing with term structure models as an example, we can construct many models from the same

[15] Change of independent variable is essentially aesthetic in latent variable models. However, change of dependent variables has real implications for the solution.

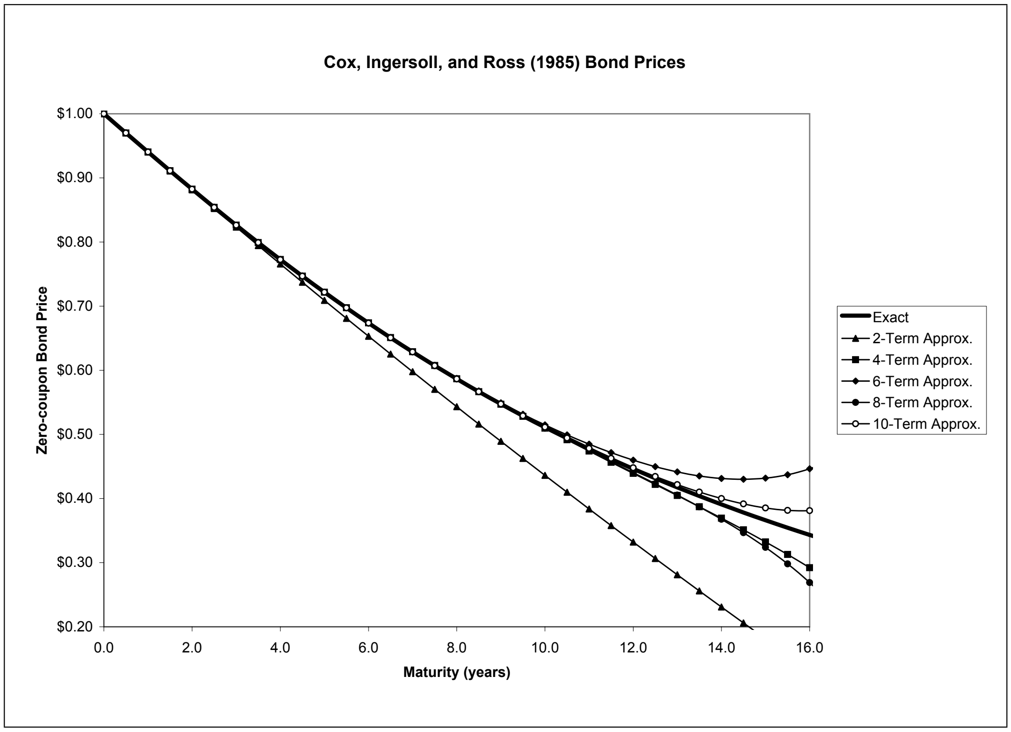

Figure 3: This figure shows prices of zero-coupon bonds as a function of maturity, from the model of Vasicek (1977), both in closed-form and with approximations including varying numbers of terms. The risk-neutral parameters are from Cheridito, Filipović, and Kimmel (2007), and the current instantaneous interest rate is taken to be $r = 6\%$. As shown, the two-term approximate price quickly deviates substantially from the true price as maturity increases beyond a few years. However, approximations with more terms do much better; the error in yield of the 10-term approximation for a bond with maturity of ten years is approximately one basis point. For this model, bond price approximations converge for all maturity; however, the convergence is not uniform, and might therefore be slow for very large maturities.

general PDE that underlies Vasicek (1977) and Ahn, Dittmar, and Gallant (2002):

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left[\frac{b^2}{2}(y-a)^2 + d\right] h(\Delta, y)$$

A final condition $h(0, y) = g(y)$ that satisfies the smoothness and growth conditions of either Theorem 2 or Corollary 2 can be used to construct a bond pricing model. By reversing the change of variables used in Section 4.1 to convert a general scalar PDE into the canonical form, bond prices, defined as:

$$P(\Delta, y) = \frac{h(\Delta, y)}{g(y)}$$

then solve:

$$\frac{\partial P}{\partial \Delta}(\Delta, y) = \frac{g'(y)}{g(y)}\frac{\partial P}{\partial y}(\Delta, y) + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}(\Delta, y) - \left[\frac{b^2}{2}(y-a)^2 + d - \frac{g''(y)}{2g(y)}\right] P(\Delta, y)$$
$$P(0, y) = 1$$

so that the solution $P(\Delta, y)$ is the price of a bond with underlying state variable dynamics (under risk-neutral probabilities):

$$dY_t = \frac{g'(Y_t)}{g(Y_t)} dt + dW_t^Q$$

with an interest rate specification:[16]

$$r(Y_t) = \frac{b^2}{2}(Y_t - a)^2 + d - \frac{g''(Y_t)}{2g(Y_t)}$$

If $g(y)$ is strictly positive (or strictly negative) for real values of y, the process Y_t can take on any real value; if $g(y)$ has zeros for real y, then there are boundaries away from $\pm\infty$.

Compared with the relative sparsity of models for which bond prices are known in closed-form, there are many specifications of $g(y)$ that allow approximation, although it may be more convenient to consider the function $w(y)$, which relates to $g(y)$ as follows:

$$g(y) = e^{\frac{b}{2}(y-a)^2} w(y)$$

For example, if $w(y)$ is a polynomial, exponential function, a polynomial multiplied by an exponential function, or a sum of all three of these types of functions, then it specifies a term structure models in which bond prices are everywhere analytic, since the final condition satisfies the conditions of Corollary 2. Functions such as $\exp(ky^2)$ satisfy the conditions of Theorem 2, so that bond prices are analytic in some region that includes the origin (and the smaller k is, the larger the region). Such functions can be multiplied by polynomials or exponential functions, added together, etc., to specify still more non-affine term structure models in which bond prices can be approximated by convergent series for some range of maturities.

Even some functions that may appear at first glance to be non-analytic also satisfy the corollary conditions; for example:

$$g(y) = e^{\frac{b}{2}(y-a)^2} \cosh\left(ky^{\frac{3}{2}}\right)$$

Despite the appearance of the fractional exponent, this condition is uniquely defined and analytic in y, since the hyperbolic cosine function is even; analyticity is evident from the power series expansion. This specification could be viewed as an intermediate case between the model of Vasicek (1977), in which $w(y)$ grows at a rate proportional to $\exp(ky)$ for some k, and the model of Ahn, Dittmar, and Gallant (2002), in which the corresponding quantity grows at a rate proportional to $\exp(ky^2)$ for some k. Here, $w(y)$ grows at a rate proportional to $\exp(ky^{3/2})$ for some k. As such, Corollary 2 establishes analyticity of the solution for all

[16] The equivalence of the PDE problem and the probabilistic problem must be verified for specific choices of $g(y)$.

values of Δ. The first few terms in the power series expansion of $h(\Delta, y)$ around $\Delta = 0$ are given by:

$$a_0(y) = e^{\frac{b}{2}(y-a)^2}\cosh\left(ky^{\frac{3}{2}}\right)$$

$$a_1(y) = e^{\frac{b}{2}(y-a)^2}\frac{1}{8}\left[(9ky+4b-8d)\cosh\left(ky^{\frac{3}{2}}\right) + 3k\left(4by^2-4aby+1\right)\frac{\sinh\left(ky^{\frac{3}{2}}\right)}{\sqrt{y}}\right]$$

$$a_2(y) = e^{\frac{b}{2}(y-a)^2}\left[\begin{array}{l} \frac{1}{128}\left[\begin{array}{l} 144b^2k^2y^3 + 9k^2\left(9k^2-32ab^2\right)y^2 - \frac{9k^2}{y} \\ +144k^2\left(3b+a^2b^2-d\right)y + 8\left(2(b-2d)^2 - 27abk^2\right) \end{array}\right]\cosh\left(ky^{\frac{3}{2}}\right) \\ +\frac{3k}{128}\left[\begin{array}{l} 72bk^2y^4 + 8b\left(10b-8d-9ak^2\right)y^3 \\ +2\left(32abd - 48ab^2 + 27k^2\right)y^2 \\ +16\left(b+a^2b^2-d\right)y + 8ab + \frac{3}{y} \end{array}\right]\frac{\sinh\left(ky^{\frac{3}{2}}\right)}{y^{\frac{3}{2}}} \end{array}\right]$$

This specification can be combined with others; for example, it can be multiplied by a polynomial, an exponential function, or a function of the form $\exp\left(ky^2\right)$, added to other such functions, etc.

With a little creativity, one can easily choose specifications of $g(y)$ that lead to some interesting features in the implied term structure model. For example, the specifications of the $g(y)$ function corresponding to both the Vasicek (1977) and Ahn, Dittmar, and Gallant (2002) is of the form:

$$g(y) = e^{k(y-\phi)^2}$$

A function of this form can be multiplied by an analytic function that grows more slowly in y; the term structure model specified by the resulting $g(y)$ then has similar interest rate behavior to these models for extreme values of the state variable; at intermediate values, the state variable dynamics will be somewhat different than those of these two models. In other words, local deformations in the drift and the diffusion functions can be introduced, while retaining the dynamic behavior of either of these two models for extreme interest rates.

So far, other than the model of Cox, Ingersoll, and Ross (1985), the cases we have considered are all based on a canonical form PDE which meets the conditions of Theorem 1 or 2 (or the corresponding corollaries). However, it is also possible to construct non-affine term structure models based on Theorem 3 or Corollary 3. Given the PDE:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left(\frac{a}{y^2} + \frac{b^2}{2}y^2 + d\right)h(\Delta, y)$$

Given some function $g_2(y)$ that satisfies the conditions of Theorem 3 or Corollary 3 (with $g_1 = 0$), we can reverse the change of variables used to construct the canonical PDE. Then bond prices are given by:

$$P(\Delta, y) = \frac{h(\Delta, y)}{y^{\frac{1+\sqrt{1+8a}}{2}}g_2(y)}$$

The bond price function then satisfies:

$$\frac{\partial P}{\partial \Delta}(\Delta, y) = \left[\frac{\alpha}{y} + \frac{g_2'(y)}{g_2(y)}\right]\frac{\partial P}{\partial y}(\Delta, y) + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}(\Delta, y) - \left[\frac{b^2}{2}y^2 + d - \frac{\alpha}{y}\frac{g_2'(y)}{g_2(y)} - \frac{1}{2}\frac{g_2''(y)}{g_2(y)}\right]P(\Delta, y)$$
$$P(0, y) = 1$$

Thus, every function $g_2(y)$ that satisfies the conditions of either Theorem 3 or Corollary 3 implicitly specifies a term structure model. We have already seen the choice of $g_2(y)$ that gives rise to the model of Cox, Ingersoll, and Ross (1985); other choices give rise to non-affine models. But in all cases, bond prices are analytic in a region of maturities including the origin, and can therefore be approximated with power series.

5.5. Other Applications

The applications we have discussed all concern the problem of pricing zero-coupon bonds in term structure models; nearly all extant models with closed-form bond prices are covered by the results of Section 4, so that bond prices can be approximated with a convergent power series. However, many additional models that have not previously appeared in the literature are also covered by these results.

Although a natural application, pricing of non-defaultable bonds is not the only problem that can be solved using our results. For example, pricing of credit derivatives requires modeling not only the interest rate process, but a default process. Given an interest rate process r_t and default intensity process λ_t, pricing of defaultable bonds involves evaluation of expressions such as:

$$d = \mathrm{E}\left[e^{-\int_t^T (r_u + \lambda_u)du}\right]$$

where the expectation is taken under a risk-neutral probability measure. Evaluation of this quantity is difficult for non-affine specifications of the r_t and λ_t processes. However, our methods establish convergence of a power series representation of this quantity for many non-affine models, greatly expanding the class of models that can be considered in practice.

Other applications are also possible. For example, Jarrow, Li, Liu, and Wu (2006) use the methods of Sectionsec:AnalyticSolutions (specifically, Theorem 3) to price callable bonds. Furthermore, for any arbitrary diffusion process, our results establish an infinite-dimensional family of moment conditions that can be approximated with power series, and that can therefore be used in an estimation procedure.

6. Conclusion

We have developed a method for closed-form approximation of conditional moments and bond prices for a wide variety of diffusion problems, and derived conditions to establish a minimum range of convergence of the approximations. Our method can be augmented by time transformation methods, and thus sometimes extend the range of convergence to include time horizons up to positive infinity, that is, uniform convergence irrespective of time horizon. These methods make feasible the rapid calculation of bond prices for many models in which such calculation would otherwise not be practical, and therefore make feasible estimation techniques for non-affine models based on likelihood or minimum distance searches. Kimmel (2008b) and Kimmel (2008a) have further examined the performance of our technique, and found rapid convergence of bond price approximations for very long maturities for many models in the literature, and also for many non-linear models that have not previously appeared. Jarrow, Li, Liu, and Wu (2006) have applied our technique to the problem of pricing callable corporate bonds in a structure model.

Potential future work includes extension of the method to multivariate diffusions. For some multivariate cases (e. g., independent state variable processes that enter additively into the interest rate function), the pricing or conditional moment problem can be broken into several independent univariate problems. Nonetheless, in the general multivariate case, it is not even possible to express the pricing PDE in the canonical form in all cases. However, the method of change of dependent variable can also lead to construction of new term structure models in which (after the change of variable) the pricing PDE is the same as the PDE that arises in multivariate affine models. Since expectations of polynomials of affine diffusions are analytic in the time horizon, at least a partial characterization of the final conditions with analytic moments is possible; as in the univariate case, each final condition with analytic moments corresponds to a term structure model with analytic bond prices. Other potential future work includes expanding the class of diffusions and interest rate specifications for which the class of problems with analytic (in time) solutions can be characterized explicitly. Other possible avenues of future research include development of methods for approximating conditional moments or asset prices that have singularities at a time horizon of zero (such as standard put and call options); if the nature of the singularity is sufficiently well understood, it may be possible to develop a series with an initial term that captures the singularity, such that the difference between the true (but unknown) solution and the initial term is analytic in the time variable, and can then be approximated by a power series. Such methods remain to be explored in full detail, however.

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A. Appendix: Proofs

This appendix includes proofs of the theorems and corollaries in the main text, as well as several auxiliary lemmas (with proofs) not included in the main text.

A.1. Proof of Theorem 1

We express the solution to the PDE as an integral over a fundamental solution:

$$h(\Delta, y) = e^{\frac{a^2\Delta^3}{6} - (ay+d)\Delta} \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{g\left(y - \frac{a\Delta^2}{2} + u\sqrt{\Delta}\right) + g\left(y - \frac{a\Delta^2}{2} - u\sqrt{\Delta}\right)}{2} du \tag{A.1}$$

The integrand is even in $\sqrt{\Delta}$, so it does not matter (for $\Delta \neq 0$) which square root is chosen. It must be demonstrated that $h(\Delta, y)$ is well-defined, is analytic in Δ and y, and solves the PDE (4.6) with final condition (4.7).

To show the existence of $h(\Delta, y)$ in the specified region, we first note that the integrand satisfies the bound:

$$\begin{aligned} \left| \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{g\left(y - \frac{a\Delta^2}{2} + u\sqrt{\Delta}\right) + g\left(y - \frac{a\Delta^2}{2} - u\sqrt{\Delta}\right)}{2} \right| &\leq \frac{ce^{-\frac{u^2}{2}}}{\sqrt{2\pi}} e^{\frac{\left(\left\|y - \frac{a\Delta^2}{2}\right\| + |u| \left\|\sqrt{\Delta}\right\|\right)^2}{2}} \\ &= \frac{ce^{\frac{u^2\left(\left\|\sqrt{\Delta}\right\|^2 - 1\right) + 2|u| \left\|y - \frac{a\Delta^2}{2}\right\| \left\|\sqrt{\Delta}\right\| + \left\|y - \frac{a\Delta^2}{2}\right\|^2}{2}}}{\sqrt{2\pi}} \end{aligned} \tag{A.2}$$

If $\left\|\sqrt{\Delta}\right\| < 1$, then the coefficient of u^2 in the exponent on the last line is negative, and the integral therefore converges for these values of Δ. The leading exponential factor in (A.1) is defined for all y and all Δ, so the function $h(\Delta, y)$ is well-defined for all $\left\|\sqrt{\Delta}\right\| < 1$.

To establish analyticity, we note that the integrand is a continuous function of u, y, and Δ. For each value of u, it is also analytic in y and Δ. This follows from the power series expansion of $g(z)$; terms which are odd in the square root of Δ from $g\left(y - a\Delta^2/2 + u\sqrt{\Delta}\right)$ cancel with the odd terms in the square root of Δ from $g\left(y - a\Delta^2/2 - u\sqrt{\Delta}\right)$. Consider the z^m term, for any $m \geq 0$, in the power series expansion of $g(z)$. Then:

$$\left(y - a\Delta^2/2 + u\sqrt{\Delta}\right)^m = \sum_{i=0}^{m} \left(y - a\Delta^2/2\right)^{m-i} \left(u\sqrt{\Delta}\right)^i \tag{A.3}$$

$$\left(y - a\Delta^2/2 - u\sqrt{\Delta}\right)^m = \sum_{i=0}^{m} \left(y - a\Delta^2/2\right)^{m-i} \left(-u\sqrt{\Delta}\right)^i \tag{A.4}$$

Since $g\left(y - a\Delta^2/2 + u\sqrt{\Delta}\right)$ and $g\left(y - a\Delta^2/2 - u\sqrt{\Delta}\right)$ are added in the integrand, terms with odd values of i cancel, and only the even terms remain. The integrand therefore is an analytic function of Δ. We also note that the integrand is uniformly bounded on compact sets of y and Δ. For any such set, simply replacing $\left\|y - a\Delta^2/2\right\|$ and $\left\|\sqrt{\Delta}\right\|$ from the right-hand side of (A.2) by their maximum values on the compact set establishes such a bound. Under these assumptions (continuity of the integrand, analyticity for each value of the variable of integration, and uniform boundedness on compacts), the integral inherits the analyticity of the integrand in y and in Δ (provided $\left\|\sqrt{\Delta}\right\| < 1$). For a discussion, see, for example, Lang (1999), who derives this result as part of what he calls "the differentiation lemma." Since the leading exponential factor on the right-hand side of (A.1) is defined and analytic for all values of y and Δ, it follows that $h(\Delta, y)$ is defined and analytic for all y and all $\left\|\sqrt{\Delta}\right\| < 1$.

Finally, we show that $h(\Delta, y)$ satisfies the partial differential equation with final condition. Satisfaction of the final condition (4.7) is straightforward; for $\Delta = 0$, we have:

$$h(0, y) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} g(y) du = g(y) \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du = g(y)$$

To show that the proposed solution solves the general partial differential equation, we calculate derivatives as follows:

$$\begin{aligned}\frac{\partial h}{\partial \Delta}(\Delta, y) &= \left[\frac{a^2\Delta^2}{2} - (ay+d)\right] h(\Delta, y) \\ &- a\Delta e^{\frac{a^2\Delta^3}{6} - (ay+d)\Delta} \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{g'\left(y - \frac{a\Delta^2}{2} + u\sqrt{\Delta}\right) + g'\left(y - \frac{a\Delta^2}{2} - u\sqrt{\Delta}\right)}{2} du \\ &+ e^{\frac{a^2\Delta^3}{6} - (ay+d)\Delta} \int_{-\infty}^{+\infty} \frac{u^2 e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{g'\left(y - \frac{a\Delta^2}{2} + u\sqrt{\Delta}\right) - g'\left(y - \frac{a\Delta^2}{2} - u\sqrt{\Delta}\right)}{4u\sqrt{\Delta}} du\end{aligned}$$

Despite the appearance of u and $\sqrt{\Delta}$ in the denominator, the integrand in the last term is well-defined when these quantities are zero. Here, $g'\left(y - a\Delta^2/2 - u\sqrt{\Delta}\right)$ is subtracted from $g'\left(y - a\Delta^2/2 + u\sqrt{\Delta}\right)$, so the terms with even i in (A.3) and (A.4) cancel, and only the terms odd in i remain. Since there is a u and a $\sqrt{\Delta}$ in the denominator, these cancel with the terms in the numerator, so the fraction is even in $u\sqrt{\Delta}$, and therefore analytic in Δ. The differentiation under the integral sign is justified by the uniform boundedness of the integrand on compact sets. The second spatial derivative is given by:

$$\begin{aligned}\frac{\partial^2 h}{\partial y^2}(\Delta, y) &= a^2\Delta^2 h(\Delta, y) \\ &- 2a\Delta e^{\frac{a^2\Delta^3}{6} - (ay+d)\Delta} \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{g'\left(y - \frac{a\Delta^2}{2} + u\sqrt{\Delta}\right) + g'\left(y - \frac{a\Delta^2}{2} - u\sqrt{\Delta}\right)}{2} du \\ &+ e^{\frac{a^2\Delta^3}{6} - (ay+d)\Delta} \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{g''\left(y - \frac{a\Delta^2}{2} + u\sqrt{\Delta}\right) + g''\left(y - \frac{a\Delta^2}{2} - u\sqrt{\Delta}\right)}{2} du\end{aligned}$$

However, the integral on the last line can be integrated by parts, which is justified by the uniform boundedness of the integrand on compacts:[17]

$$\begin{aligned}&\int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{g''\left(y - \frac{a\Delta^2}{2} + u\sqrt{\Delta}\right) + g''\left(y - \frac{a\Delta^2}{2} - u\sqrt{\Delta}\right)}{2} du \\ &= \int_{-\infty}^{+\infty} \frac{u^2 e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{g'\left(y - \frac{a\Delta^2}{2} + u\sqrt{\Delta}\right) - g'\left(y - \frac{a\Delta^2}{2} - u\sqrt{\Delta}\right)}{2u\sqrt{\Delta}} du\end{aligned}$$

Substituting these expressions for the derivatives of $h(\Delta, y)$ into the general PDE (4.6), one finds that $h(\Delta, y)$ is the solution.

[17]The bounds on the first and second derivatives of $g(z)$ follow by application of Cauchy's integral theorem.

A.2. Proof of Theorem 2

Although the expressions are more complicated, the proof proceeds similarly to the proof of Theorem 1. We express the solution to the PDE as an integral over a fundamental solution:

$$h(\Delta, y) = e^{\frac{b}{2}(y-a)^2 + \left(\frac{b}{2}-d\right)\Delta} \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\begin{bmatrix} \phi\left(a + e^{b\Delta}(y-a) + u\sqrt{\tau(\Delta)}\right) \\ +\phi\left(a + e^{b\Delta}(y-a) - u\sqrt{\tau(\Delta)}\right) \end{bmatrix}}{2} du \tag{A.5}$$

where:

$$\phi(z) = e^{-\frac{b}{2}(z-a)^2} g(z)$$

The integrand is even in $\sqrt{\tau(\Delta)}$, so it does not matter (for $\Delta = 0$, for which $\tau(0) = 0$) which square root is chosen. It must be demonstrated that $h(\Delta, y)$ is well-defined, is analytic in Δ and y, and solves the PDE (4.8) with final condition (4.9).

To show the existence of $h(\Delta, y)$ in the specified region, we first note that the integrand satisfies the bound:

$$\left| \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\begin{bmatrix} \phi\left(a + e^{b\Delta}(y-a) + u\sqrt{\tau(\Delta)}\right) \\ +\phi\left(a + e^{b\Delta}(y-a) - u\sqrt{\tau(\Delta)}\right) \end{bmatrix}}{2} \right| \leq \frac{ce^{-\frac{u^2}{2}}}{\sqrt{2\pi}} e^{\frac{\left(\left\|a+e^{b\Delta}(y-a)\right\| + |u| \left\|\sqrt{\tau(\Delta)}\right\|\right)^2}{2}}$$

$$= \frac{ce^{\frac{u^2\left(\left\|\sqrt{\tau(\Delta)}\right\|^2 - 1\right) + 2|u| \left\|a+e^{b\Delta}(y-a)\right\| \left\|\sqrt{\tau(\Delta)}\right\| + \left\|a+e^{b\Delta}(y-a)\right\|^2}{2}}}{\sqrt{2\pi}}$$

Since the coefficient on u^2 in the exponent on the last line is negative whenever $\left\|\sqrt{\tau(\Delta)}\right\| < 1$, the integral converges for these values. Since the leading exponential factor in (A.5) is defined for all Δ and y, it follows that $h(\Delta, y)$ is well-defined for all complex y and Δ such that $\left\|\sqrt{\tau(\Delta)}\right\| < 1$.

To establish analyticity, we note that the integrand is a continuous function of u, y, and τ. For each value of u, it is also analytic in y and τ; as in the proof of Theorem 1, this follows from the power series expansion of $\phi(z)$. But $\tau(\Delta)$ is an analytic function of Δ, so the integrand is also analytic in Δ. It is also uniformly bounded on compact sets of y and Δ. Then, by the differentiation lemma (see Lang (1999)), the integral inherits the analyticity of the integrand in y and in Δ (provided $\left\|\sqrt{\tau(\Delta)}\right\| < 1$).

Finally, we show that $h(\Delta, y)$ satisfies the partial differential equation with final condition. Satisfaction of the final condition (4.9) is straightforward; for $\Delta = 0$, we have:

$$h(0, y) = e^{\frac{b}{2}(y-a)^2} \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \phi(y)\, du = g(y) \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du = g(y)$$

To show that the proposed solution solves the general partial differential equation, we calculate derivatives as follows. Note that all derivations remain valid in the special case of $b = 0$, in which case $\tau(\Delta) = \Delta$.

The derivative with respect to Δ is given by:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \left(\frac{b}{2} - d\right) h(\Delta, y)$$

$$+ be^{b\Delta}(y-a)\, e^{\frac{b}{2}(y-a)^2 + \left(\frac{b}{2}-d\right)\Delta} \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\begin{bmatrix} \phi'\left(a + e^{b\Delta}(y-a) + u\sqrt{\tau(\Delta)}\right) \\ +\phi'\left(a + e^{b\Delta}(y-a) - u\sqrt{\tau(\Delta)}\right) \end{bmatrix}}{2} du$$

$$+ e^{2b\Delta} e^{\frac{b}{2}(y-a)^2 + \left(\frac{b}{2}-d\right)\Delta} \int_{-\infty}^{+\infty} \frac{u^2 e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\begin{bmatrix} \phi'\left(a + e^{b\Delta}(y-a) + u\sqrt{\tau(\Delta)}\right) \\ -\phi'\left(a + e^{b\Delta}(y-a) - u\sqrt{\tau(\Delta)}\right) \end{bmatrix}}{4u\sqrt{\tau(\Delta)}} du$$

The presence of u and $\sqrt{\tau(\Delta)}$ in the denominator on the last line pose no problems, by reasoning similar to that used in Theorem 1, the terms in the power series expansion of $\phi'\left(a + e^{b\Delta}(y-a) + u\sqrt{\tau(\Delta)}\right)$ that are even in $u\sqrt{\tau(\Delta)}$ cancel with the corresponding terms in $\phi'\left(a + e^{b\Delta}(y-a) - u\sqrt{\tau(\Delta)}\right)$. The only remaining terms are odd in $u\sqrt{\tau(\Delta)}$, and after cancellation with the denominator, the fraction in the integrand (extended by continuity to $u = 0$ and $\tau(\Delta) = 0$) is even in $u\sqrt{\tau(\Delta)}$, and therefore analytic in $\tau(\Delta)$. The integrand is also uniformly bounded on compact sets, which justifies differentiation under the integral sign. The second spatial derivative is:

$$\frac{\partial^2 h}{\partial y^2}(\Delta, y) = \left[b^2 (y-a)^2 + b\right] h(\Delta, y)$$

$$+ 2be^{b\Delta}(y-a)\, e^{\frac{b}{2}(y-a)^2 + \left(\frac{b}{2}-d\right)\Delta} \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\begin{bmatrix} \phi'\left(a + e^{b\Delta}(y-a) + u\sqrt{\tau(\Delta)}\right) \\ +\phi'\left(a + e^{b\Delta}(y-a) - u\sqrt{\tau(\Delta)}\right) \end{bmatrix}}{2} du$$

$$+ e^{2b\Delta} e^{\frac{b}{2}(y-a)^2 + \left(\frac{b}{2}-d\right)\Delta} \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\begin{bmatrix} \phi''\left(a + e^{b\Delta}(y-a) + u\sqrt{\tau(\Delta)}\right) \\ +\phi''\left(a + e^{b\Delta}(y-a) - u\sqrt{\tau(\Delta)}\right) \end{bmatrix}}{2} du$$

But by integration by parts (which is justified by the uniform boundedness of the integral):

$$\int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\begin{bmatrix} \phi''\left(a + e^{b\Delta}(y-a) + u\sqrt{\tau(\Delta)}\right) \\ +\phi''\left(a + e^{b\Delta}(y-a) - u\sqrt{\tau(\Delta)}\right) \end{bmatrix}}{2} du$$

$$= \int_{-\infty}^{+\infty} \frac{u^2 e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\begin{bmatrix} \phi'\left(a + e^{b\Delta}(y-a) + u\sqrt{\tau(\Delta)}\right) \\ -\phi'\left(a + e^{b\Delta}(y-a) - u\sqrt{\tau(\Delta)}\right) \end{bmatrix}}{2u\sqrt{\tau(\Delta)}} du$$

Substitution of these expressions into the general PDE shows that $h(\Delta, y)$ is a solution.

A.3. Proof of Corollary 1

Choose a value of k, and define the norm $\|y\| \equiv |y| / \sqrt{k}$. Then $g(y)$ satisfies the conditions of Theorem 1 for this norm and for $c = c_k$. So by Theorem 1, there exists a solution $h(\Delta, y)$ to the partial differential equation (with final condition) that is analytic for all y and for all $|\Delta| < k$. Since we can choose any $k > 0$, the circle of analyticity can be shown to be as large as desired. Furthermore, the solutions constructed in the proofs of Theorems 1 and 2 do not depend on the value of c or on the norm, so it is clear that the solutions constructed here for different values of k are in fact the same functions. Consequently, the solution is defined and analytic in y and Δ.

A.4. Proof of Corollary 2

Choose a value of k, and define the norm $\|y\| \equiv |y| / \sqrt{k}$. Then $g(y)$ satisfies the conditions of Theorem 2 for this norm and for $c = c_k$. So by Theorem 2, there exists a solution $h(\Delta, y)$ to the partial differential equation (with final condition) that is analytic for all y and for all $|\tau(\Delta)| < k$. Since we can choose any $k > 0$, the circle of analyticity can be shown to be as large as desired. Furthermore, the solutions constructed in the proofs of Theorems 2 do not depend on the value of c or on the norm, so it is clear that the solutions constructed here for different values of k are in fact the same functions. Consequently, the solution is defined and analytic in y and τ for all $\tau(\Delta)$. Since $\tau(\Delta)$ is well-defined and analytic for all Δ, it follows that $h(\Delta, y)$ is well-defined and analytic for all y and Δ.

A.5. Proof of Theorem 3

The proof of Theorem 3 is much longer and more complicated than the proofs of Theorem 1 and Theorem 2, involves explicit construction of the solution to a partial differential equation as an infinite series, and makes extensive use of integral operators. It should be noted that the lengthy construction of the PDE solution is needed only to prove the theorem and establish analyticity of the solution, not to construct its power series approximation. For the latter purpose, those willing to trust in the correctness of the proof may ignore it completely, and need only verify that a particular problem satisfies the conditions of the theorem. Construction of the power series then proceeds by the simple recursive procedure.

The construction of the PDE solution uses four different integral operators, and Lemmas 1 through 4 establish various properties of those operators. Lemma 5 constructs a solution to a related PDE that is not in the canonical form, and finally the theorem proof uses Lemma 5 (several times) to construct the PDE solution.

A.5.1. Lemma 1

Lemma 1 *Let $\psi(z)$ be an analytic and even function defined for all complex z, and let there exist some $c > 0$ and some norm (over the reals) $|z| / \sqrt{k} \leq \|z\| \leq |z| / \sqrt{k_1}$ such that $\psi(z)$ satisfies the bound:*

$$|\psi(z)| \leq c e^{\frac{\|z\|^2}{2}}$$

Then:

$$\nu(\Delta, y) \equiv \int_0^{+\infty} \frac{2e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\psi\left(y+u\sqrt{\Delta}\right)+\psi\left(y-u\sqrt{\Delta}\right)}{2}\right] du \tag{A.6}$$

is analytic in both variables for all y and all $\left\|\sqrt{\Delta}\right\| < 1$, and is also an even function of y. Furthermore, $\nu(\Delta, y)$ satisfies the bound:

$$|\nu(\Delta, y)| \leq \frac{2ce^{\frac{\|y\|^2}{2\left(1-\|\sqrt{\Delta}\|^2\right)}}}{\sqrt{1-\left\|\sqrt{\Delta}\right\|^2}} \tag{A.7}$$

The spatial derivative of $\nu(\Delta, y)$ is given by:

$$\frac{\partial \nu}{\partial y}(\Delta, y) = \int_0^{+\infty} \frac{2ue^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\psi\left(y+u\sqrt{\Delta}\right)-\psi\left(y-u\sqrt{\Delta}\right)}{2\sqrt{\Delta}}\right] du \tag{A.8}$$

Finally, $\nu(\Delta, y)$ satisfies:

$$\frac{\partial \nu}{\partial \Delta}(\Delta, y) - \frac{1}{2}\frac{\partial^2 \nu}{\partial y^2}(\Delta, y) = 0 \tag{A.9}$$

$$\nu(0, y) = \psi(y) \tag{A.10}$$

for all complex y and $\left\|\sqrt{\Delta}\right\| < 1$.

The integral constructed in the lemma statement is the same as the construction in the proof of Theorem 2, which establishes the existence of the integral and its analyticity for all y and $\left\|\sqrt{\Delta}\right\| < 1$. The proof of Theorem 2 also establishes that $\nu(\Delta, y)$ solves (A.9) with final condition (A.10). It remains only to show that $\nu(\Delta, y)$ is even in y, that is satisfies (A.7), and that it has spatial derivative (A.8). Evenness of the integrand in y follows from the evenness of $\psi(z)$ (which was not assumed in Theorem 2); the integral then inherits the evenness of the integrand. From the bound on $\psi(z)$ and the properties of a norm:

$$\left|\frac{\psi\left(y+u\sqrt{\Delta}\right)+\psi\left(y-u\sqrt{\Delta}\right)}{2}\right| \leq ce^{\frac{\left(\|y\|+u\left\|\sqrt{\Delta}\right\|\right)^2}{2}} \tag{A.11}$$

The bound on $\nu(\Delta, y)$ follows by substituting the right-hand side of (A.11) into (A.6) and performing the integration. The uniform convergence of the integral on compacts (as discussed in the proof of Theorem 2) justifies differentiating under the integral and integration by parts, so the spatial derivative can be found as follows:

$$\begin{aligned}
\frac{\partial \nu}{\partial y}(\Delta, y) &= \int_0^{+\infty} \frac{2e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\partial}{\partial y}\left[\frac{\psi\left(y+u\sqrt{\Delta}\right)+\psi\left(y-u\sqrt{\Delta}\right)}{2}\right] du \\
&= \int_0^{+\infty} \frac{2e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\partial}{\partial u}\left[\frac{\psi\left(y+u\sqrt{\Delta}\right)-\psi\left(y-u\sqrt{\Delta}\right)}{2\sqrt{\Delta}}\right] du \\
&= \int_0^{+\infty} \frac{2ue^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\psi\left(y+u\sqrt{\Delta}\right)-\psi\left(y-u\sqrt{\Delta}\right)}{2\sqrt{\Delta}}\right] du
\end{aligned}$$

The last step follows by integration by parts.

A.5.2. Lemma 2

Lemma 2 *Let $\psi(z)$ be an analytic and even function defined for all complex z, and let there exist some $c > 0$ and some norm (over the reals) $|z|/\sqrt{k_2} \leq \|z\| \leq |z|/\sqrt{k_1}$ such that $\psi(z)$ satisfies the bound:*

$$|\psi(z)| \leq c e^{\frac{\|z\|^2}{2}}$$

Then:

$$\phi(\Delta, y) \equiv \int_0^{+\infty} \frac{2u^2 e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\psi\left(y + u\sqrt{\Delta}\right) - \psi\left(y - u\sqrt{\Delta}\right)}{2yu\sqrt{\Delta}} \right] du \tag{A.12}$$

is analytic in both variables for all y and all $\left\|\sqrt{\Delta}\right\| < 1$, and is also an even function of y. Furthermore, $\phi(\Delta, y)$ satisfies the bound:

$$|\phi(\Delta, y)| \leq \frac{2c\left(e^2+1\right)}{k_1} \frac{e^{\frac{\|y\|^2}{2\left(1-\left\|\sqrt{\Delta}\right\|^2\right)}}}{\left(1 - \left\|\sqrt{\Delta}\right\|^2\right)^{\frac{3}{2}}} \tag{A.13}$$

The integrand is continuous, and, for each value of u, analytic in y and Δ. This follows from the power series expansion of $\psi(\tau, z)$, which contains only even terms in z; substituting in $z = y \pm u\sqrt{\Delta}$ and expanding with the binomial theorem, the numerator contains only terms which have odd powers of y and of $u\sqrt{\Delta}$, so, after dividing by the denominator, only even powers remain. It follows from the bound on $\psi(z)$ and the properties of a norm that:

$$\left| \frac{\psi\left(y + u\sqrt{\Delta}\right) - \psi\left(y - u\sqrt{\Delta}\right)}{2yu\sqrt{\Delta}} \right| \leq \frac{c}{|y|\, u \left|\sqrt{\Delta}\right|} e^{\frac{\left(\|y\| + u\left\|\sqrt{\Delta}\right\|\right)^2}{2}} \leq \frac{c}{k_1 \|y\|\, u \left\|\sqrt{\Delta}\right\|} e^{\frac{\left(\|y\| + u\left\|\sqrt{\Delta}\right\|\right)^2}{2}} \tag{A.14}$$

However, we can also establish another bound on this expression using the maximum modulus principle of analytic functions. Specifically, we show that:

$$\left| \frac{\psi\left(y + u\sqrt{\Delta}\right) - \psi\left(y - u\sqrt{\Delta}\right)}{2yu\sqrt{\Delta}} \right| \leq \frac{2ce}{k_1} e^{\frac{\left(\|y\| + u\left\|\sqrt{\Delta}\right\|\right)^2}{2}} \tag{A.15}$$

If $\|y\|\, u \left\|\sqrt{\Delta}\right\| \geq 1/2$, this follows directly from (A.14). If $\|y\|\, u \left\|\sqrt{\Delta}\right\| < 1/2$ and $\|y\| \geq 1/\sqrt{2}$, it follows by replacing $u\left\|\sqrt{\Delta}\right\|$ in (A.14) by $1/\left(2\|y\|\right)$, since the integrand is an analytic function of $u\sqrt{\Delta}$, since $\left\|u\sqrt{\Delta}\right\| < 1/\left(2\|y\|\right)$, and since an analytic function takes its maximum value in a closed set on the boundary of that set. The bound is established similarly when $u\|y\| \left\|\sqrt{\Delta}\right\| < 1/2$ and $u\left\|\sqrt{\Delta}\right\| \geq 1/\sqrt{2}$; in this case, $\|y\|$ in (A.14) is replaced by $1/\left(2u\left\|\sqrt{\Delta}\right\|\right)$. Finally, when $\|y\| < 1/\sqrt{2}$ and $u\left\|\sqrt{\Delta}\right\| < 1/\sqrt{2}$, the bound follows by replacing each of these quantities by $1/\sqrt{2}$.

Convergence of the integral for $\left\|\sqrt{\Delta}\right\| < 1$ follows from (A.15). Furthermore, the integrand is continuous, for each value of u, analytic in y and Δ, and uniformly bounded on compacts (simply replace $\|y\|$ and $\left\|\sqrt{\Delta}\right\|$ in (A.15) by their maximum values on the compact set). It follows (see, for example, Lang (1999)) that the integral inherits the analyticity of the integrand. Evenness in y follows from the evenness of the integrand in y, which itself follows directly from the evenness of the $\psi(z)$ function.

The only property that remains to be established is satisfaction of the bound (A.13). However, unlike the other properties, for which the bound (A.15) suffices, both (A.14) and (A.15) are needed. Specifically, we divide the range of integration in (A.12) into two pieces, $0 \leq u \leq 1/\left(\|y\| \left\|\sqrt{\Delta}\right\|\right)$ and $1/\left(\|y\| \left\|\sqrt{\Delta}\right\|\right) \leq u \leq +\infty$, apply the bound from (A.15) in the first part, and apply the bound from (A.14) in the second part. It follows that:

$$\left|\int_0^{\frac{1}{\|y\|\|\sqrt{\Delta}\|}} \frac{2u^2 e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\psi\left(y+u\sqrt{\Delta}\right)-\psi\left(y-u\sqrt{\Delta}\right)}{2yu\sqrt{\Delta}}\right] du\right| \leq \frac{2ce^2}{k_1} \frac{e^{\frac{\|y\|^2}{2}}}{\left(1-\left\|\sqrt{\Delta}\right\|^2\right)^{\frac{3}{2}}}$$

$$\left|\int_{\frac{1}{\|y\|\|\sqrt{\Delta}\|}}^{+\infty} \frac{2u^2 e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\psi\left(y+u\sqrt{\Delta}\right)-\psi\left(y-u\sqrt{\Delta}\right)}{2yu\sqrt{\Delta}}\right] du\right| \leq \frac{2c}{k_1} \frac{e^{\frac{\|y\|^2}{2\left(1-\|\sqrt{\Delta}\|^2\right)}}}{\left(1-\left\|\sqrt{\Delta}\right\|^2\right)^{\frac{3}{2}}}$$

The required bound (A.13) follows by summing these two.

A.5.3. Lemma 3

Lemma 3 *Let $\psi(\tau, z)$ be defined and analytic for all complex z and $\|\tau\| < 1$ for some norm (over the reals) $|z|/\sqrt{k_2} \leq \|z\| \leq |z|/\sqrt{k_1}$, and let $\psi(\tau, z)$ be even in z. Let there exist some $c > 0$ and some integer $n \geq 0$ such that $\psi(\tau, z)$ satisfies the bound:*

$$|\psi(\tau, z)| \leq c\left[-\ln\left(1-\left\|\sqrt{\tau}\right\|^2\right)\right]^n \frac{e^{\frac{\|z\|^2}{2\left(1-\|\sqrt{\tau}\|^2\right)}}}{\left(1-\|\sqrt{\tau}\|^2\right)^{\frac{3}{2}}}$$

Then:

$$\nu(\Delta, y) \equiv \Delta \int_0^1 \int_0^{+\infty} \frac{2e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\psi\left(s\Delta, y+u\sqrt{(1-s)\Delta}\right)+\psi\left(s\Delta, y-u\sqrt{(1-s)\Delta}\right)}{2}\right] duds \qquad \text{(A.16)}$$

is analytic in both variables for all y and all $\left\|\sqrt{\Delta}\right\| < 1$, and is also an even function of y. Furthermore, $\nu(\Delta, y)$ satisfies the bound:

$$|\nu(\Delta, y)| \leq \frac{2c}{n+1}\left[-\ln\left(1-\left\|\sqrt{\Delta}\right\|^2\right)\right]^{n+1} \frac{e^{\frac{\|y\|^2}{2\left(1-\|\sqrt{\Delta}\|^2\right)}}}{\sqrt{1-\left\|\sqrt{\Delta}\right\|^2}} \qquad \text{(A.17)}$$

The spatial derivative of $\nu(\Delta, y)$ is given by:

$$\frac{\partial \nu}{\partial y}(\Delta, y) = \Delta \int_0^1 \int_0^{+\infty} \frac{2ue^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\psi\left(s\Delta, y+u\sqrt{(1-s)\Delta}\right)-\psi\left(s\Delta, y-u\sqrt{(1-s)\Delta}\right)}{2\sqrt{(1-s)\Delta}}\right] duds$$

Finally, $\nu(\Delta, y)$ satisfies:

$$\frac{\partial \nu}{\partial \Delta}(\Delta, y) - \frac{1}{2}\frac{\partial^2 \nu}{\partial y^2}(\Delta, y) = \psi(\Delta, y) \qquad \text{(A.18)}$$

$$\nu(0, y) = 0 \qquad \text{(A.19)}$$

for all complex y and $\left\|\sqrt{\Delta}\right\| < 1$.

From the bound on $\psi(\tau, z)$ and the properties of a norm, we have:

$$\left| \frac{\begin{bmatrix} \psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) \\ +\psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right) \end{bmatrix}}{2} \right| \le c \left[-\ln\left(1 - s\left\|\sqrt{\Delta}\right\|^2\right)\right]^n \frac{e^{\frac{\left(\|y\| + u\sqrt{1-s}\left\|\sqrt{\Delta}\right\|\right)^2}{2\left(1-s\left\|\sqrt{\Delta}\right\|^2\right)}}}{\left(1 - s\left\|\sqrt{\Delta}\right\|^2\right)^{\frac{3}{2}}} \tag{A.20}$$

Since s takes values in $[0,1]$, the integral converges for any $\left\|\sqrt{\Delta}\right\| < 1$. The integrand is also continuous, and for each value of s and u, analytic in y and Δ. This follows from the power series expansion of $\psi(\tau, z)$; the terms that are odd in the square root of $(1-s)\Delta$ cancel. Furthermore, the integrand is uniformly bounded on compact sets of y and Δ. The integral is therefore analytic in y and Δ as well. Evenness of the integral in y follows immediately from evenness of the integrand, which itself is a consequence of the evenness of $\psi(\tau, z)$ in z. The bound (A.17) follows by substituting the left-hand side of (A.20) in for the bracketed expression in (A.16), and performing the integration.

The uniform convergence of the integrand on compacts justifies operations such as differentiation under the integral sign, and integration by parts. The spatial derivative then follows by integration by parts:

$$\begin{aligned} \frac{\partial \nu}{\partial y}(\Delta, y) &= \Delta \int_0^1 \int_0^{+\infty} \frac{2e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\partial}{\partial y} \left[\frac{\psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) + \psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)}{2} \right] duds \\ &= \Delta \int_0^1 \int_0^{+\infty} \frac{2e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\partial}{\partial u} \left[\frac{\psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) - \psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)}{2\sqrt{(1-s)\Delta}} \right] duds \\ &= \Delta \int_0^1 \int_0^{+\infty} \frac{2ue^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) - \psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)}{2\sqrt{(1-s)\Delta}} \right] duds \end{aligned}$$

It remains to show that (A.18) and (A.19) are satisfied. The time derivative can be found as follows:

$$\begin{aligned} \frac{\partial \nu}{\partial \Delta}(\Delta, y) &= \begin{bmatrix} \Delta \int_0^1 \int_0^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\partial}{\partial \Delta} \left[\psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) + \psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)\right] duds \\ + \int_0^1 \int_0^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) + \psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)\right] duds \end{bmatrix} \\ &= \begin{bmatrix} \int_0^1 \int_0^{+\infty} \frac{ue^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\partial}{\partial u} \left[\frac{\psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) + \psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)}{2(1-s)} \right] duds \\ + \int_0^1 \int_0^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) + \psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)\right] duds \\ + \int_0^1 \int_0^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} s\frac{\partial}{\partial s} \left[\psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) + \psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)\right] duds \end{bmatrix} \\ &= \begin{bmatrix} \int_0^1 \int_0^{+\infty} \frac{ue^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\partial}{\partial u} \left[\frac{\psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) + \psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)}{2(1-s)} \right] duds \\ + \psi(\Delta, y) \end{bmatrix} \end{aligned} \tag{A.21}$$

The last step follows by integrating the middle integral by parts; one of the two resulting terms evaluates to $\phi(\Delta, y)$, and the other cancels with the first integral. The second spatial derivative is:

$$
\begin{aligned}
\frac{\partial^2 \nu}{\partial y^2}(\Delta, y) &= \Delta \int_0^1 \int_0^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\partial^2}{\partial y^2} \left[\psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) + \psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)\right] duds \\
&= \int_0^1 \int_0^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\partial^2}{\partial u^2} \left[\frac{\psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) + \psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)}{(1-s)}\right] duds \qquad \text{(A.22)} \\
&= \int_0^1 \int_0^{+\infty} \frac{ue^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{\partial}{\partial u} \left[\frac{\psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) + \psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)}{(1-s)}\right] duds
\end{aligned}
$$

The last step follows from integration by parts. Summing (A.21) and (A.22), we find that (A.18) is satisfied. Satisfaction of (A.19) follows simply by evaluating (A.16) at $\Delta = 0$.

A.5.4. Lemma 4

Lemma 4 *Let $\psi(\tau, z)$ be defined and analytic for all complex z and $\|\tau\| < 1$ for some norm (over the reals) $|z|/\sqrt{k_2} \le \|z\| \le |z|/\sqrt{k_1}$, and let $\psi(\tau, z)$ be even in z. Let there exist some $c > 0$ and some integer $n \ge 0$ such that $\psi(\tau, z)$ satisfies the bound:*

$$
|\psi(\tau, z)| \le c\left[-\ln\left(1 - \left\|\sqrt{\tau}\right\|^2\right)\right]^n \frac{e^{\frac{\|z\|^2}{2\left(1-\|\sqrt{\tau}\|^2\right)}}}{\left(1 - \left\|\sqrt{\tau}\right\|^2\right)^{\frac{3}{2}}} \qquad \text{(A.23)}
$$

Then:

$$
\phi(\Delta, y) \equiv \Delta \int_0^1 \int_0^{+\infty} \frac{2u^2 e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) - \psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)}{2yu\sqrt{(1-s)\Delta}}\right] duds \qquad \text{(A.24)}
$$

is analytic in both variables for all y and all $\left\|\sqrt{\Delta}\right\| < 1$, and is also an even function of y. Furthermore, $\phi(\Delta, y)$ satisfies the bound:

$$
|\phi(\Delta, y)| \le \frac{2c\left(e^2+1\right)k_2}{k_1(n+1)} \left[-\ln\left(1 - \left\|\sqrt{\Delta}\right\|^2\right)\right]^{n+1} \frac{e^{\frac{\|y\|^2}{2\left(1-\|\sqrt{\Delta}\|^2\right)}}}{\left(1 - \left\|\sqrt{\Delta}\right\|^2\right)^{\frac{3}{2}}}
$$

The integrand in (A.24) is continuous, and, for each value of u and s, analytic in y and Δ. This follows by from the power series expansion of $\psi(\tau, z)$ (which contains only even terms in z). Substituting in the arguments for z and expanding using the binomial theorem, the numerator contains only odd powers of y and $u\sqrt{(1-s)\Delta}$, which cancel with the denominator, leaving only even powers. From (A.23) and the properties of a norm:

$$
\left|\frac{\left[\begin{array}{c}\psi\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) \\ -\psi\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)\end{array}\right]}{2yu\sqrt{(1-s)\Delta}}\right| \le \frac{c\left[-\ln\left(1 - s\left\|\sqrt{\Delta}\right\|^2\right)\right]^n}{k_1\|y\|u\sqrt{1-s}\left\|\sqrt{\Delta}\right\|} \frac{e^{\frac{\|y\|+u\sqrt{(1-s)}\|\sqrt{\Delta}\|^2}{2\left(1-s\|\sqrt{\Delta}\|^2\right)}}}{\left(1 - s\left\|\sqrt{\Delta}\right\|^2\right)^{\frac{3}{2}}}
$$

Proceeding as in the proof of Lemma 2, application of the maximum modulus principle gives another bound:

$$\left| \frac{\begin{bmatrix} \psi\left(s\Delta, y+u\sqrt{(1-s)\,\Delta}\right) \\ -\psi\left(s\Delta, y-u\sqrt{(1-s)\,\Delta}\right) \end{bmatrix}}{2yu\sqrt{(1-s)\,\Delta}} \right| \leq \frac{2ce\left[-\ln\left(1-s\left\|\sqrt{\Delta}\right\|^2\right)\right]^n}{k_1} \frac{e^{\frac{\|y\|+u\sqrt{(1-s)}\|\sqrt{\Delta}\|^2}{2\left(1-s\|\sqrt{\Delta}\|^2\right)}}}{\left(1-s\left\|\sqrt{\Delta}\right\|^2\right)^{\frac{3}{2}}} \tag{A.25}$$

Then by breaking the inner integral in (A.24) into two, and applying (A.25) on $u \in \left[0, 1/\left(\|y\|\sqrt{(1-s)\,\Delta}\right)\right]$, and (A.25) on $u \in \left[1/\left(\|y\|\sqrt{(1-s)\,\Delta}\right), +\infty\right)$ establishes the required bound.

A.5.5. Lemma 5

Lemma 5 *Let $\psi(y)$ be an analytic and even function defined for all complex y, and let $\eta(\Delta, y)$ be an analytic function, even in y, defined for all y and Δ such that $\left\|\sqrt{\tau(\Delta)}\right\| < 1$, where $|z|/\sqrt{k_2} \leq \|z\| \leq |z|/\sqrt{k_1}$ is a norm over the reals. Further let there exist some $c > 0$ and $d > 0$ such that $\psi(z)$ and $\eta(\tau, z)$ satisfy the bounds:*

$$|\psi(z)| \leq ce^{\frac{\|z\|^2}{2}}$$
$$|\eta(\Delta, z)| \leq de^{\frac{\|z\|^2}{2\left(1-\left\|\sqrt{\tau(\Delta)}\right\|^2\right)}}$$

Then for any complex γ, there exists a function $w(\Delta, y)$ that is defined and analytic for all complex y and Δ such that $\left\|\sqrt{\tau(\Delta)}\right\| < 1$, even in y, and that satisfies:

$$y\frac{\partial w}{\partial \Delta}(\Delta, y) = \gamma\left[\frac{\partial w}{\partial y}(\Delta, y) + y\eta(\Delta, y)\right] + \frac{y}{2}\frac{\partial^2 w}{\partial y^2}(\Delta, y) - y\left(\frac{b^2}{2}y^2 + d\right)w(\Delta, y) \tag{A.26}$$
$$w(0, y) = \psi(y) \tag{A.27}$$

Furthermore, there exist continuous functions $d_1(\Delta)$ and $d_2(\Delta)$, defined for all $\left\|\sqrt{\tau(\Delta)}\right\| < 1$, such that $w(\Delta, y)$ and its first spatial derivative satisfy the bounds:

$$|w(\Delta, y)| \leq d_1(\Delta)\, e^{\frac{\left\|e^{b\Delta}y\right\|^2}{2\left(1-\left\|\sqrt{\tau(\Delta)}\right\|^2\right)}} \tag{A.28}$$

$$\left|\frac{\partial w}{\partial y}(\Delta, y)\right| \leq d_2(\Delta)\, |y|\, e^{\frac{\left\|e^{b\Delta}y\right\|^2}{2\left(1-\left\|\sqrt{\tau(\Delta)}\right\|^2\right)}} \tag{A.29}$$

Proof:

We construct the solution explicitly, by the parametrix method of Levi (1907), as described in Friedman (1964). Note, however, that the assumptions of Friedman (1964) are not satisfied in this case, because the coefficients of the PDE are not bounded. We therefore must modify the method.

First, define:

$$\nu_0(\Delta, y) \equiv \int_0^{+\infty} \frac{2e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\psi\left(y + u\sqrt{\Delta}\right) + \psi\left(y - u\sqrt{\Delta}\right)}{2} \right] du \tag{A.30}$$

$$\phi_1(\Delta, y) \equiv \eta(\Delta, y) + \int_0^{+\infty} \frac{2u^2 e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\psi\left(y + u\sqrt{\Delta}\right) - \psi\left(y - u\sqrt{\Delta}\right)}{2yu\sqrt{\Delta}} \right] du$$

Then for each integer $i \geq 1$, define:

$$\nu_i(\Delta, y) \equiv \Delta \int_0^1 \int_0^{+\infty} \frac{2e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\phi_i\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) + \phi_i\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)}{2} \right] duds$$

$$\phi_{i+1}(\Delta, y) \equiv \Delta \int_0^1 \int_0^{+\infty} \frac{2u^2 e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\phi_i\left(s\Delta, y + u\sqrt{(1-s)\Delta}\right) - \phi_i\left(s\Delta, y - u\sqrt{(1-s)\Delta}\right)}{2yu\sqrt{(1-s)\Delta}} \right] duds$$

Finally, for each $i \geq 0$, define:

$$\xi_i(\Delta, y) = e^{\frac{b}{2}\left(y^2 + \Delta\right) + (b\gamma - d)\Delta} \nu_i\left(e^{b\Delta} y, \tau(\Delta)\right)$$

The solution to the PDE is then given by:

$$w(\Delta, y) \equiv \xi_0(\Delta, y) + \sum_{n=1}^{\infty} \gamma^n \xi_n(\Delta, y) \tag{A.31}$$

Four things must be proven. First, it must be shown that $w(\Delta, y)$ is well-defined. Second, it must be shown that $w(\Delta, y)$ is analytic in Δ and y, and even in y. Third, it must be shown that it solves the PDE with final condition. Finally, it must be shown that $w(\Delta, y)$ and its spatial derivative satisfy the stated bounds.

To show existence of the integrals and convergence of the infinite sum, we first note that $\psi(y)$ satisfies the conditions of Lemmas 1 and 2. Together with the assumptions on $\eta(\Delta, y)$, it follows by these lemmas that both $\nu_0(\Delta, y)$ and $\phi_1(\Delta, y)$ are defined and analytic for all y and Δ such that $\left\|\sqrt{\Delta}\right\| < 1$, and are also even in y. The lemma also establishes bounds on $\nu_0(\Delta, y)$ and the second term in the definition of $\phi_1(\Delta, y)$; together with the assumptions on $\eta(\Delta, y)$, we have:

$$|\nu_0(\Delta, y)| \leq \frac{2ce^{\frac{\|y\|^2}{2\left(1 - \|\sqrt{\Delta}\|^2\right)}}}{\sqrt{1 - \left\|\sqrt{\Delta}\right\|^2}}$$

$$|\phi_1(\Delta, y)| \leq \left(d + \frac{2c\left(e^2 + 1\right)}{k_1}\right) \frac{e^{\frac{\|y\|^2}{2\left(1 - \|\sqrt{\Delta}\|^2\right)}}}{\left(1 - \left\|\sqrt{\Delta}\right\|^2\right)^{\frac{3}{2}}}$$

But $\phi_1(\Delta, y)$ itself then satisfies the conditions of Lemmas 3 and 4. Lemma 4 can thus be applied recursively, with each $\phi_i(\Delta, y)$ for $i \geq 1$ satisfying the conditions of the lemma, and the construction in the lemma producing $\phi_{i+1}(\Delta, y)$, which itself satisfies the conditions of both Lemmas 3 and 4. Proceeding in this way, by repeated application of this lemma, it is established that for all values of $i \geq 1$, the functions $\phi_i(\Delta, y)$ and $\nu_i(\Delta, y)$ are defined and analytic in both variables (with $\left\|\sqrt{\Delta}\right\| < 1$), even in y, and satisfy the bounds:

$$|\phi_i(\Delta, y)| \leq \left(d + \frac{2c\left(e^2+1\right)}{k_1}\right) \frac{e^{\frac{\|y\|^2}{2\left(1-\left\|\sqrt{\Delta}\right\|^2\right)}}}{\left(1-\left\|\sqrt{\Delta}\right\|^2\right)^{\frac{3}{2}}} \frac{\left[-\frac{2c\left(e^2+1\right)k_2 \ln\left(1-\left\|\sqrt{\Delta}\right\|^2\right)}{k_1}\right]^{(i-1)}}{(i-1)!} \tag{A.32}$$

$$|\nu_i(\Delta, y)| \leq \left(\frac{dk_1}{k_2\left(e^2+1\right)} + \frac{c}{k_2}\right) \frac{e^{\frac{\|y\|^2}{2\left(1-\left\|\sqrt{\Delta}\right\|^2\right)}}}{\sqrt{1-\left\|\sqrt{\Delta}\right\|^2}} \frac{\left[-\frac{2\left(e^2+1\right)k_2 \ln\left(1-\left\|\sqrt{\Delta}\right\|^2\right)}{k_1}\right]^{i}}{i!} \tag{A.33}$$

Note that the $\nu_i(\Delta, y)$ are bounded by the coefficients of an exponential power series, and therefore converge uniformly on compact subsets of y and Δ, provided $\left\|\sqrt{\Delta}\right\| < 1$. It follows immediately that the $\xi_i(\Delta, y)$ are also analytic in both y and Δ, and converge uniformly on compact subsets of y and Δ, where $\left\|\sqrt{\tau(\Delta)}\right\| < 1$. Since each term in the sum that defines $w(\Delta, y)$ is analytic for all y and $\left\|\sqrt{\tau(\Delta)}\right\|$, uniform convergence on compacts establishes the analyticity of the sum. It is also readily evident that $w(\Delta, y)$ is even in y, since each term in the sum is even in y.

It remains to establish that $w(\Delta, y)$ is a solution to the PDE. Satisfaction of the final condition is established in a straightforward way. For every $i \geq 1$, $\nu_i(0, y) = 0$. From (A.30), it follows by application of Lemma 1 that $\nu_0(0, y) = \psi(y)$, which establishes the final condition.

To show that $w(\Delta, y)$ is a solution to the general PDE, we first note that:

$$\begin{aligned}
&\frac{\partial w}{\partial \Delta}(\Delta, y) - \frac{1}{2}\frac{\partial^2 w}{\partial \Delta^2}(\Delta, y) + \left(\frac{b^2}{2}y^2 + d\right) w(\Delta, y) \\
&= e^{\frac{b}{2}\left(y^2+\Delta\right)+(b\gamma-d)} \left[\begin{array}{c} b\gamma\nu_0\left(\tau(\Delta), e^{b\Delta}y\right) + e^{2b\Delta}\left[\begin{array}{c} \nu_0^{(\tau)}\left(\tau(\Delta), e^{b\Delta}y\right) \\ -\frac{1}{2}\nu_0^{(zz)}\left(\tau(\Delta), e^{b\Delta}y\right)\end{array}\right] \\ + \sum_{n=1}^{\infty}\gamma^n \left(b\gamma\nu_n\left(\tau(\Delta), e^{b\Delta}y\right) + e^{2b\Delta}\left[\begin{array}{c} \nu_n^{(\tau)}\left(\tau(\Delta), e^{b\Delta}y\right) \\ -\frac{1}{2}\nu_n^{(zz)}\left(\tau(\Delta), e^{b\Delta}y\right)\end{array}\right]\right)\end{array}\right] \\
&= b\gamma w(\Delta, y) + e^{\frac{b}{2}\left(y^2+\Delta\right)+(b\gamma-d)} e^{2b\Delta} \sum_{n=1}^{\infty} \gamma^n \phi_n\left(\tau(\Delta), e^{b\Delta}y\right)
\end{aligned} \tag{A.34}$$

where $\nu_i^{(\tau)}(\tau, z)$ denotes partial differentiation with respect to the first argument, and $\nu_i^{(z)}(\tau, z)$ and $\nu_i^{(zz)}(\tau, z)$ denote partial differentiation with respect to the second argument once and twice, respectively. The differentiation term-by-term in (A.34) is justified by the uniform convergence of (A.31) on compacts. The evaluation of the derivatives comes from Lemmas 1 and 3. Next, we note that:

$$\begin{aligned}\frac{\partial w}{\partial y}(\Delta,y) &= e^{\frac{b}{2}\left(y^2+\Delta\right)+(b\gamma-d)}\left(\begin{bmatrix} by\nu_0\left(\tau(\Delta),e^{b\Delta}y\right) \\ +e^{b\Delta}\nu_0^{(z)}\left(\tau(\Delta),e^{b\Delta}y\right)\end{bmatrix}+\sum_{n=1}^{\infty}\gamma^n\begin{bmatrix} by\nu_n\left(\tau(\Delta),e^{b\Delta}y\right) \\ +e^{b\Delta}\nu_n^{(z)}\left(\tau(\Delta),e^{b\Delta}y\right)\end{bmatrix}\right) \\ &= byw\left(\tau(\Delta),e^{b\Delta}y\right)+e^{b\Delta}e^{\frac{b}{2}\left(y^2+\Delta\right)+(b\gamma-d)}\begin{pmatrix} y\left[\phi_1\left(\tau(\Delta),e^{b\Delta}y\right)-\eta\left(\tau(\Delta),e^{b\Delta}y\right)\right] \\ +\sum_{n=1}^{\infty}\gamma^n y\phi_{n+1}\left(\tau(\Delta),e^{b\Delta}y\right)\end{pmatrix}\end{aligned} \tag{A.35}$$

By substituting (A.34) and (A.35) into (A.26), it can be seen that $w(\Delta,y)$ satisfies the general PDE.

The bounds on $w(\Delta,y)$ and its derivative follow immediately when one recognizes that, except for $\nu_0(\Delta,y)$, the bounds in (A.33) are the coefficients of an exponential power series. It follows that:

$$|w(\Delta,y)| \leq \left(\frac{dk_1}{k_2\left(e^2+1\right)}+\frac{2c}{\min\left(1,k_2\right)}\right)e^{\left|\frac{b}{2}\left(y^2+\Delta\right)+(b\gamma-d)\Delta\right|}\frac{e^{\frac{\left\|e^{b\Delta}y\right\|^2}{2\left(1-\left\|\sqrt{\tau(\Delta)}\right\|^2\right)}}}{\sqrt{1-\left\|\sqrt{\tau(\Delta)}\right\|^2}}\left(1-\left\|\sqrt{\tau(\Delta)}\right\|^2\right)^{-\frac{2|\gamma|\left(e^2+1\right)k_2}{k_1}}$$

which establishes (A.28). (A.29) follows similarly, by applying the bound on $\phi_i(\Delta,y)$ from (A.32) to (A.35).

A.5.6. Proof of Theorem 3

The general PDE of Lemma 5 (i. e., (A.26) and (A.27)) is not in the canonical form (4.2). However, the general PDE of the theorem (i. e., (4.12), (4.13), and (4.14)) can be converted to the PDE of the lemma by a change of variables, and we can therefore express the solution of this PDE in terms of the solution to the PDE of the Lemma.

We begin with the case $\left(\sqrt{1+8a}\right)/2 \notin \mathbb{N}$. We first apply Lemma 5 with $\gamma = \left(1-\sqrt{1+8a}\right)/2$, $\psi(y) = g_1(y)$, and $\eta(\Delta,y) = 0$, and find a solution $h_1(\Delta,y) = w(\Delta,y)$. By applying Lemma 5 again with $\gamma = \left(1+\sqrt{1+8a}\right)/2$, and $\psi(y) = g_2(y)$, we find another solution $h_2(\Delta,y) = w(\Delta,y)$. We now can construct a solution to the original PDE:

$$h(\Delta,y) = y^{\frac{1-\sqrt{1+8a}}{2}}h_1(\Delta,y)+y^{\frac{1+\sqrt{1+8a}}{2}}h_2(\Delta,y)$$

By inspection, this solution satisfies the PDE with final condition (i. e., (4.12), (4.13), and (4.14)).

For $\left(\sqrt{1+8a}\right)/2 \in \mathbb{N}$, we apply Lemma 5 with $\gamma = \left(1-\sqrt{1+8a}\right)/2$, $\psi(y) = g_1(y)$, and $\eta(\Delta,y) = 0$, and then again with $\gamma = \left(1+\sqrt{1+8a}\right)/2$, $\psi(y) = g_2(y)$, and $\eta(\Delta,y) = 0$, obtaining solutions $h_1(\Delta,y)$ and $h_2(\Delta,y)$, respectively. We then apply Lemma 5 a third time, with $\gamma = \left(1-\sqrt{1+8a}\right)/2$, $\psi(y) = 0$, and:

$$\eta(\Delta,y) = -y^{-1+\sqrt{1+8a}}\frac{\partial h_2}{\partial y}(\Delta,y)-\frac{\sqrt{1+8a}}{2}y^{-2+\sqrt{1+8a}}h_2(\Delta,y)$$

Since $h_2(\Delta,y)$ is itself the result of an application of Lemma 5, it is analytic in both variables, even in y, and satisfies the boundedness conditions in the lemma statement, (A.26) and (A.27). It follows that $\eta(\Delta,y)$ is analytic in both variables, and even in y. The function $h_2(\Delta,y)$ is analytic and even in y, so its derivative with respect to y is analytic and odd in y. Since $\sqrt{1+8a}$ is a non-negative even integer, the derivative of $h_2(\Delta,y)$ is premultiplied by y raised to a power that is either -1 or a positive odd integer. If it is a positive odd integer, the first term is clearly analytic and even in y; if it is -1, then the result of dividing an analytic and odd function by y is analytic (and even) everywhere except $y = 0$, but extends by analytic continuation to this value. (This can be seen, for example, from the power series of the derivative of $h_2(\Delta, y)$, and then term-by-term division by y.) If $\sqrt{1+8a} = 0$, then the second term is zero, and is trivially analytic in y; if $\sqrt{1+8a}$ is some positive even integer, then the second term is the analytic function $h_2(\Delta, y)$ multiplied by a constant and a non-negative even power of y, which is also analytic. So both terms are analytic and even in y.

In general, this $\eta(\Delta, y)$ in the third application of Lemma 5 does *not* satisfy the conditions of the lemma for the same norm $\|y\|$ used in the first two applications. However, for any $0 < \epsilon < 1$, we define the norm $\|y\|_\epsilon \equiv \|y\| / (1-\epsilon)$. Then, for any such ϵ, $\eta(\Delta, y)$ satisfies the conditions of Lemma 5 for some d. Applying the lemma a third time and denoting the result by $h_3(\Delta, y)$, we construct the solution to the PDE as:

$$h(\Delta, y) = [h_1(\Delta, y) + h_3(\Delta, y)]\, y^{\frac{1-\sqrt{1+8a}}{2}} + h_2(\Delta, y)\, y^{\frac{1+\sqrt{1+8a}}{2}} \ln y$$

By inspection, this solution solves the general PDE with final condition. The construction of $h_3(\Delta, y)$ is valid only for $\left\|\sqrt{\tau(\Delta)}\right\|_\epsilon < 1$, not $\left\|\sqrt{\tau(\Delta)}\right\| < 1$. But ϵ can be chosen to be arbitrarily small, so analyticity of $h_3(\Delta, y)$ for any particular value of $\left\|\sqrt{\tau(\Delta)}\right\| < 1$ can be established by applying Lemma 5 with the norm $\|y\|_\epsilon$ for sufficiently small ϵ. The boundedness results from Lemma 5 then do not apply to $h_3(\Delta, y)$ uniformly for all $\left\|\sqrt{\tau(\Delta)}\right\| < 1$, but there are no boundedness conditions in the theorem statement to be established; the boundedness conditions in the lemma are only needed for $h_2(\Delta, y)$, constructed in the second application, to show that the third application is justified. The theorem is now proven for the case of $\sqrt{1+8a}/2 \in \mathbb{N}$.

A.6. Proof of Corollary 3

Choose a value of k, and define the norm $\|y\| \equiv |y| / \sqrt{k}$. Then $g_1(y)$ and $g_2(y)$ satisfy the conditions of Theorem 3 for this norm and for $c = c_k$. So by application of Theorem 3, there exist functions $h_1(\Delta, y)$ and $h_2(\Delta, y)$, defined and analytic for all y and $|\Delta| < k$, such that $h(\Delta, y)$, as defined in the corollary statement, satisfies the partial differential equation with final condition. Since we can choose any $k > 0$, the circle of analyticity can be shown to be as large as desired. Furthermore, the construction in the proof of Theorem 3 does not depend on c or on the choice of the norm, so it is clear that the solutions constructed for different values of k are the same function. Consequently, the solution is defined and analytic for all complex Δ.