UNIVERSITY OF WATERLOO

Faculty of Engineering

Department of Mechanical and Mechatronics Engineering

Project 3

A report prepared for:

MTE 322

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3B Mechatronics Engineering

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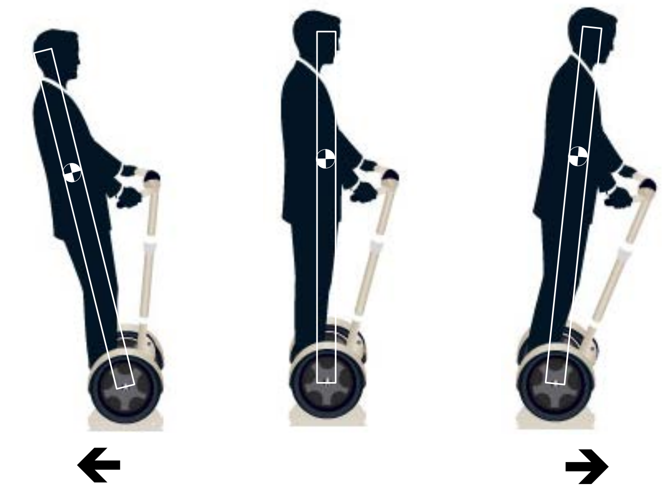
Project #3 Assigned: Mar. 27th Due: Apr. 3rd by 2:30 pm

Segway is a personal transportation device that balances by itself and can be moved by leaning forward or backward as shown in Fig. 1.



(a) Going backward (b) Staying balanced (c) Going forward

Fig. 1: The balancing ability of a Segway

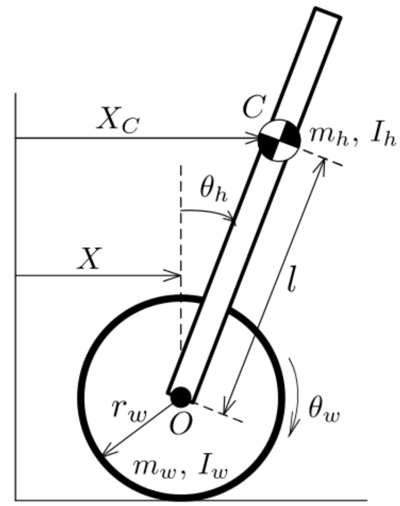


Fig. 2: Schematic

In this project, we use a simplified Segway model to go through the process of rapid prototyping of control systems, i.e. modeling, simulation and validation of controller.

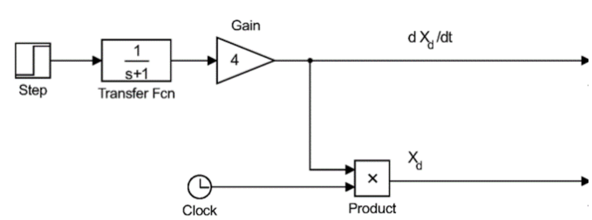
1) Derive the mathematical model of the Segway using the simplified schematic shown in Fig. 2.
   - Wheel radius: $r_w = 0.25$ m,
   - Mass of the wheel: $m_w = 20$ kg, Inertia of the wheel: $I_w = 0.1\ \text{kg}\cdot\text{m}^2$
   - Mass of the human: $m_h = 60$ kg, Inertia of the human: $I_h = 10\ \text{kg}\cdot\text{m}^2$
   - The center of mass of the human from the wheel axle: $l = 1$ m
   - The input to the system is the motor torque (T_w) applied to the wheel axle (not shown in Fig. 2).

2) Linearize the plant model from 1). Apply the following controller to the linearized model and compute the closed-loop poles for the system from X_d to X.

$$T_w = -K_1(X - X_d) - K_2(\dot{X} - \dot{X}_d) - K_3\theta_h - K_4\dot{\theta}_h$$

where X_d is the desired position and $K_1 = -70, K_2 = -250, K_3 = -1800, K_4 = -600$.

3) Download the Matlab file (system_parameters.m) and the Simulink file (Segway_plant_model.slx) from the course website. Close the loop with the controller in 2). Simulate the closed-loop response using the desired signal, $\dot{X}_d = 4(1 - e^{-t})$ m/s which can be generated by the following blocks. Add scopes to X, θ_h and T_w so that we can monitor their time responses. Set the simulation time as 10 second.



1 Derivation of the Mathematical Model

1.1 Free Body Diagrams

The system was modelled using two free body diagrams.

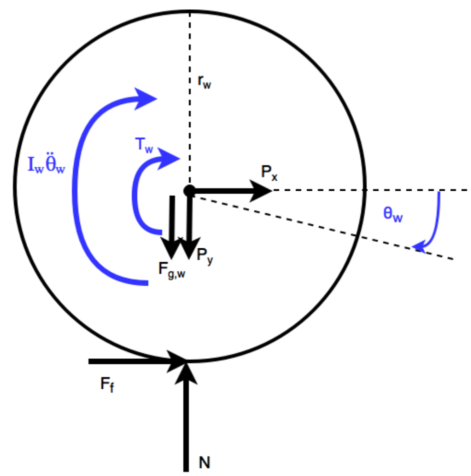


Figure 1: Free body diagram of Segway wheel

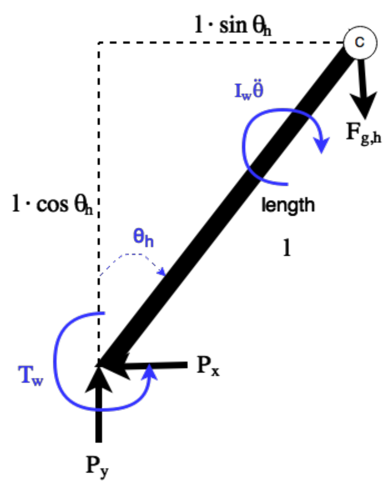


Figure 2: Free body diagram of human

1.2 Kinematic and Dynamic Equations

Linear equations of motion for the human have been found using first principals for kinematic motion.

$$X_c = (X + l\sin\theta_h)\hat{\imath}$$
$$\dot{X}_c = (\dot{X} + \dot{\theta}_h l\cos\theta_h)\hat{\imath}$$
$$\ddot{X}_c = (\ddot{X} + \ddot{\theta}_h l\cos\theta_h - \dot{\theta}_h^2 l\sin\theta_h)\hat{\imath} \quad (1)$$

$$Y_c = (l\cos\theta_h)\hat{\jmath}$$
$$\dot{Y}_c = (-\dot{\theta}_h l\sin\theta_h)\hat{\jmath}$$
$$\ddot{Y}_c = (\ddot{\theta}_h l\sin\theta_h + \dot{\theta}_h^2 l\cos\theta_h)\hat{\jmath} \quad (2)$$

Angular equations of motion for the human and wheel are derived from summation of moments. The wheel equation is shown below, the $F_f r_w$ term is moment induced from surface friction on the wheel.

$$I_w \ddot{\theta}_w = T_w - F_f r_w$$

Solving for friction yields

$$F_f = \frac{T_w - I_w \ddot{\theta}_w}{r_w} \; (3)$$

Angular equation of motion for the human is:

$$I_h \ddot{\theta}_h = -T_w + P_y l \sin\theta_h + P_x l \cos\theta_h \;\; (4)$$

A summation of forces on the human and wheel may be taken to derive a linear equation of motion.

$$m_w \ddot{X} = P_x + F_f \; (5)$$
$$m_w \ddot{Y} = N - P_y - F_{g,w} = 0 \; (6)$$

Linear equations of motion for the human is provided below. The derivation of $\ddot{X}_c$ (1) may be substituted into this equation.

$$-m_h \ddot{X}_c = P_x$$
$$-m_h\left(\ddot{X} + \ddot{\theta}_h l \cos\theta_h + \dot{\theta}_h^2 l \sin\theta_h\right) = P_x \; (7)$$

1.3 Deriving Coupled Differential Equations

Using $\hat{n} = -\cos\theta_h \,\hat{\imath} + \sin\theta_h \,\hat{\jmath}$ transformation the normal linear motion (1) & (2) to derive *ma* on the LHS, and the RHS is derived from the FBD force equilibrium, is represented below:

$$m_h\left(\ddot{X}\cos\theta_h + \ddot{\theta}_h l\right)\hat{n} = \left(-m_h g \sin\theta_h + P_x \cos\theta_h + P_y \sin\theta_h\right)\hat{n} \; (8)$$

Substitute equation (8) into equation (4). Solving,

$$I_h \ddot{\theta}_h + T_w = l\left(P_x \cos\theta_h + P_y \sin\theta_h\right)$$
$$I_h \ddot{\theta}_h + T_w = l m_h\left(\ddot{X}\cos\theta_h + \ddot{\theta}_h l - g \sin\theta_h\right)$$
$$I_h \ddot{\theta}_h + T_w = m_h l \cos\theta_h \ddot{X} + m_h l^2 \ddot{\theta}_h - m_h l g \sin\theta_h$$
$$(m_h l \cos\theta_h)\ddot{X} - (I_h - m_h l^2)\ddot{\theta}_h + m_h g l \sin\theta_h = T_w \; (9)$$

The above is the first of two coupled equations. Next, substituting (3) & (7) into (5) to derive the second equation below,

$$m_w \ddot{X} = P_x + F_f$$
$$m_w \ddot{X} = P_x + \left(\frac{T_w - I_w \ddot{\theta}_w}{r_w}\right)$$
$$m_w \ddot{X} = -m_h\left(\ddot{X} + \ddot{\theta}_h l \cos\theta_h - \dot{\theta}_h^2 l \sin\theta_h\right) + \left(\frac{T_w - I_w \ddot{\theta}_w}{r_w}\right)$$
$$(m_w + m_h) r_w \ddot{X} = -m_h r_w l \cos\theta_h \ddot{\theta}_h + m_h r_w l \sin\theta_h \dot{\theta}_h^2 + T_w - I_w \ddot{\theta}_w$$
$$r_w(m_w + m_h)\ddot{X} + (m_h r_w l \cos\theta_h)\ddot{\theta}_h - (m_h r_w l \sin\theta_h)\dot{\theta}_h^2 + I_w \ddot{\theta}_w = T_w \; (10)$$

The differential equations must be in terms of X and θ_h. Thus, θ_w may be substituted by $\frac{X}{r_w}$ into (10).

$$r_w(m_w + m_h)\ddot{X} + (m_h r_w l\cos\theta_h)\ddot{\theta}_h - (m_h r_w l\sin\theta_h)\dot{\theta}_h^2 + \frac{I_w}{r_w}\ddot{X} = T_w$$
$$\left(r_w(m_w + m_h) + \frac{I_w}{r_w}\right)\ddot{X} + (m_h r_w l\cos\theta_h)\ddot{\theta}_h - (m_h r_w l\sin\theta_h)\dot{\theta}_h^2 = T_w$$

Thus, the coupled second order differential equations are:

$$\boldsymbol{(m_h l\cos\theta_h)\ddot{X} - \left(I_h - m_h l^2\right)\ddot{\theta}_h + m_h g l\sin\theta_h = T_w \; (11)}$$
$$\boldsymbol{\left(r_w(m_w + m_h) + \frac{I_w}{r_w}\right)\ddot{X} + (m_h r_w l\cos\theta_h)\ddot{\theta}_h - (m_h r_w l\sin\theta_h)\dot{\theta}_h^2 = T_w \; (12)}$$

2 Linearization and Closed Loop Poles

2.1 Linearization of the Differential Equations

To begin, coupled second order transfer functions are simplified using the small angle approximation:

$$\sin\theta = \theta, \quad \cos\theta = 1$$
$$(m_h l)\ddot{X} - (I_h - m_h l^2)\ddot{\theta}_h + (m_h g l)\theta_h = T_w \; (13)$$
$$\left(r_w(m_w + m_h) + \frac{I_w}{r_w}\right)\ddot{X} + (m_h r_w l)\ddot{\theta}_h - (m_h r_w l\theta_h)\dot{\theta}_h^2 = T_w \; (14)$$

To linearize the second differential equation, a Taylor Series estimate is used:

$$f(x) = f(0) + f'(0) + \frac{f''(0)}{2!} + \cdots$$

However, the first two terms will be considered a sufficient estimation. The initial conditions are assumed to be $\theta_h(0) = 0$, $\dot{\theta}_h(0) = 0$, and $\ddot{\theta}_h(0) = 0$ Thus,

$$(m_h r_w l\theta_h)\dot{\theta}_h^2 = (m_h r_w l\theta_h)\dot{\theta}_h^2 + m_h r_w l\left(\dot{\theta}_h^3 + 2\theta_h\dot{\theta}_h\ddot{\theta}_h\right)$$
$$(m_h r_w l\theta_h)\dot{\theta}_h^2 = 0$$

Thus, the final linearized differential equations in the time domain are:

$$\boldsymbol{(m_h l)\ddot{X} - \left(I_h - m_h l^2\right)\ddot{\theta}_h + (m_h g l)\theta_h = T_w} \; (15)$$
$$\boldsymbol{\left(r_w(m_w + m_h) + \frac{I_w}{r_w}\right)\ddot{X} + (m_h r_w l)\ddot{\theta}_h = T_w} \; (16)$$

2.2 Determination of the Transfer Function

The Laplace transform is performed on both equations (15) & (16) to analyze them in the s domain. The initial conditions are assumed to be zero.

$$\boldsymbol{s^2 m_h l X(s) - \left(s^2\left(I_h - m_h l^2\right) + (m_h g l\right)\theta_h(s) = T_w(s)} \; (17)$$
$$\boldsymbol{s^2\left(r_w(m_w + m_h) + \frac{I_w}{r_w}\right)X(s) + s^2 m_h r_w l\theta_h(s) = T_w(s)} \; (18)$$

The torque, T_w, may be represented in the s domain as:

$$T_w = -K_1(X - X_d) - K_2(\dot{X} - \dot{X}_d) - K_3\theta_h - K_4\dot{\theta}_h$$
$$\boldsymbol{T_w(s) = (-K_1 - sK_2)X(s) + (K_1 + sK_2)X_d(s) - (K_3 + sK_4)\theta_h(s)} \quad (19)$$

Substituting (19) into (17) and (18) using MATLAB ode solver to yield the following transfer function:

$$G(s) = \frac{X(s)}{X_d(s)} = \frac{25\,(\,4375s^3 + 1225s^2 + 73575s + 20601)}{1500s^4 + 187625s^3 + 1010468s^2 - 1839375s - 515025}$$

2.3 Determination of Closed Loop Poles

The MATLAB function poles( ) was used to determine the Closed Loop poles of the system. The resulting poles are provided in below.

Table 1: Closed Loop Poles of the Segway

Poles
1.63179503407219028220843604186 44
-0.247812802042445661229017217743 06
-7.11398880763113734334460854960 09
-119.353326757731940610968143607 85

3 Closed loop response simulation

The file is uploaded on learn.

4 Contributions

4.1 Max Pfeifle

Max tackled the derivation for the mathematical model of the system.

4.2 Matthew Van Heukelom

Matt did the FDB in draw.io. He also tackled the linearization section and helped Max with derivation of the mathematical model.

4.3 Pavel Shering

Pavel completed the MATLAB portion of the report, and helped Max when he got stuck in the derivation of the mathematical model. Pavel also helped with the report.

5 Appendix A – MATLAB Code

```
%% Linearization and Closed-loop poles

clear all;
clc;

mw = 20; % [kg] wheel mass
Iw = 0.1; % [kg*m^2] inertia of the wheel
mh = 60; % [kg] humna mass
Ih = 10; % [kg*m^2] inertia of the wheel

rw = 0.25; % wheel radius
l = 1; % [m] center of mass of the human form the wheel axis
g = 9.81; % [kg/m^2]
bw = 0; %

K = [-70, -250, -1800, -600];

syms s theta_h X Xd Tw

Tw = (-K(1) -s*K(2))*X + (K(1)+s*K(2))*Xd - (K(3)+s*K(4))*theta_h;
ode1 = (s^2)*(mh*l)*X - (s^2*(Ih - mh*l^2) - mh*g*l)*theta_h  == Tw;
ode2 = s^2*(rw*(mw+mh)+Iw/rw)*X + s^2*mh*rw*l*theta_h == Tw;

sol = solve(ode1,theta_h)==solve(ode2,theta_h)
TF = simplifyFraction(solve(sol,X)/Xd)
poles(TF,s)
```