Project #2 Second Order ODE Applications

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This report involves modelling vibration of a spring-mass-damper mechanical system through manipulation of one-dimensional motion of mass differential equation (1).

$$mx'' + cx' + kx = F(t) \tag{1}$$

Part A

Converting the mass motion differential equation (1) into an equation in terms of natural frequency and damping ratio.

$\omega_n = \sqrt{\frac{k}{m}}$ - natural frequency vibration without damping $\quad x(0) = x_0$

$\zeta = \frac{c}{2m\omega_n}$ - damping ratio, indicates transient system response $\quad x'(0) = x'_0$

$$\begin{aligned} x'' + \frac{c}{m}x' + \frac{k}{m}x &= \frac{F(t)}{m}, \qquad \frac{k}{m} = \omega_n^2 \\ x'' + \frac{c}{m}x' + \omega_n^2 x &= \frac{F(t)}{m}, \qquad \frac{c}{m} = 2\zeta\omega_n \\ \mathbf{x'' + 2\zeta\omega_n x' + \omega_n^2 x} &= \mathbf{\frac{F(t)}{m}} \end{aligned} \tag{2}$$

Part B

$x'' + 2\zeta\omega_n x' + \omega_n^2 x = \frac{F(t)}{m} \qquad F(t) = F_0 sin(\Omega t) \qquad \Omega$ - frequency of forcing function

$x'' + 2\zeta\omega_n x' + \omega_n^2 x = \frac{F_0}{m} sin(\Omega t) \qquad x(0) = x'(0) = 0$

$$\therefore x(t) = x_c + x_p \tag{3}$$

x_c - transient solution, homogeneous, reaches equilibrium due to damping
x_p - steady state solution

Steady State Solution

$$x_c = e^{-\zeta\omega_n t}\left(Acos(\sqrt{1-\zeta^2}\,\omega_n t) + Bsin(\sqrt{1-\zeta^2}\,\omega_n t)\right) \tag{4}$$

Solve for steady state - assume solution form

(5)

$$x_p = Acos(\Omega t) + Bsin(\Omega t)$$

- assume $\Omega \neq \omega_n$
- sub $x_p(3)$ into ODE(1)

$$\begin{aligned} x_p' &= -\Omega Asin(\Omega t) + \Omega Bcos(\Omega t) \\ &= \Omega\left(-Asin(\Omega t) + Bcos(\Omega t)\right) \\ x_p'' &= \Omega\left(-\Omega Acos(\Omega t) - \Omega Bsin(\Omega t)\right) \\ &= -\Omega^2\left(Acos(\Omega t) + Bsin(\Omega t)\right) \end{aligned}$$

$$-\Omega^2\left(Acos(\Omega t) + Bsin(\Omega t)\right) + 2\zeta\omega_n\Omega\left(-Asin(\Omega t) + Bcos(\Omega t)\right) + \omega_n^2(Acos(\Omega t) + Bsin(\Omega t) = \frac{F_0}{m}sin(\Omega t)$$

- collect likes and split into sine and cosine components

(6)

$$\begin{aligned} -\Omega^2 Bsin(\Omega t) - 2\zeta\omega_n\Omega Asin(\Omega t) + \omega_n^2 Bsin(\Omega t) &= \frac{F_0}{m}sin(\Omega t) \\ \left(B(\omega_n^2 - \Omega^2) - 2\zeta\omega_n\Omega A\right)sin(\Omega t) &= \frac{F_0}{m}sin(\Omega t) \end{aligned}$$

(7)

$$\begin{aligned} -\Omega^2 Acos(\Omega t) + 2\zeta\omega_n\Omega Bcos(\Omega t) + \omega_n^2 Acos(\Omega t) &= \frac{F_0}{m}sin(\Omega t) \\ \left(A(\omega_n^2 - \Omega^2) + 2\zeta\omega_n\Omega B\right)cos(\Omega t) &= 0 \end{aligned}$$

- rearrange cosine (7) to solve for A

(8)

$$A = \frac{-2\zeta\omega_n\Omega B}{(\omega_n^2 - \Omega^2)}$$

- sub (8) into (6) to solve for B

(9)

$$\begin{aligned} B(\omega_n^2 - \Omega^2) - 2\zeta\omega_n\Omega\frac{(-2\zeta\omega_n\Omega B)}{(\omega_n^2 - \Omega^2)} &= \frac{F_0}{m} \\ \frac{B[(\omega_n^2 - \Omega^2)^2 + (2\zeta\omega_n\Omega)^2]}{(\omega_n^2 - \Omega^2)} &= \frac{F_0}{m} \\ \mathbf{B} = \frac{\frac{F_0}{m}(\omega_n^2 - \Omega^2)}{(\omega_n^2 - \Omega^2)^2 + (2\zeta\omega_n\Omega)^2} \end{aligned}$$

- use solution of B (9) to solve for A (8)

(10)

$$\begin{aligned} \therefore A &= \frac{\frac{F_0}{m}(\omega_n^2 - \Omega^2)(-2\zeta\omega_n\Omega)}{[(\omega_n^2 - \Omega^2)^2 + (2\zeta\omega_n\Omega)^2](\omega_n^2 - \Omega^2)} \\ \mathbf{A} &= \frac{\frac{F_0}{m}(-2\zeta\omega_n\Omega)}{(\omega_n^2 - \Omega^2)^2 + (2\zeta\omega_n\Omega)^2} \end{aligned}$$

(11)

$$\therefore \mathbf{x_p} = \frac{\frac{F_0}{m}(-2\zeta\omega_n\Omega)}{(\omega_n^2 - \Omega^2)^2 + (2\zeta\omega_n\Omega)^2}\mathbf{cos(\Omega t)} + \frac{\frac{F_0}{m}(\omega_n^2 - \Omega^2)}{(\omega_n^2 - \Omega^2)^2 + (2\zeta\omega_n\Omega)^2}\mathbf{sin(\Omega t)}$$

- sub x_p (11) and x_c (4) into (3) and solve for A & B using initial conditions

(12)

$$
\begin{aligned}
x(t) &= e^{-\zeta\omega_n t} A cos(\sqrt{1-\zeta^2}\omega_n t) + e^{-\zeta\omega_n t} B sin(\sqrt{1-\zeta^2}\omega_n t) + \\
&\frac{\frac{F_0}{m}(-2\zeta\omega_n\Omega)}{(\omega_n^2-\Omega^2)^2+(2\zeta\omega_n\Omega)^2} cos(\Omega t) + \frac{\frac{F_0}{m}(\omega_n^2-\Omega^2)}{(\omega_n^2-\Omega^2)^2+(2\zeta\omega_n\Omega)^2} sin(\Omega t) \\
x(t) &= 0 \\
0 &= e^{-\zeta\omega_n(0)} A(1) + \frac{\frac{F_0}{m}(-2\zeta\omega_n\Omega)}{(\omega_n^2-\Omega^2)^2+(2\zeta\omega_n\Omega)^2}(1) \\
\mathbf{A} &= \frac{\frac{\mathbf{F_0}}{\mathbf{m}}(\mathbf{2}\zeta\omega_{\mathbf{n}}\mathbf{\Omega})}{(\omega_{\mathbf{n}}^{\mathbf{2}}-\mathbf{\Omega^2})^{\mathbf{2}}+(\mathbf{2}\zeta\omega_{\mathbf{n}}\mathbf{\Omega})^{\mathbf{2}}}
\end{aligned}
$$

(13)

$$
\begin{aligned}
x'(t) &= 0 \\
x'(t) &= -\zeta\omega_n e^{-\zeta\omega_n t} A cos(\sqrt{1-\zeta^2}\omega_n t) - e^{-\zeta\omega_n t} A(\sqrt{1-\zeta^2}\omega_n) sin(\sqrt{1-\zeta^2}\omega_n t) - \\
&\zeta\omega_n e^{-\zeta\omega_n t} B sin(\sqrt{1-\zeta^2}\omega_n t) + e^{-\zeta\omega_n t} B(\sqrt{1-\zeta^2}\omega_n) cos(\sqrt{1-\zeta^2}\omega_n t) + \\
&\frac{\frac{F_0}{m}(2\zeta\omega_n\Omega)}{(\omega_n^2-\Omega^2)^2+(2\zeta\omega_n\Omega)^2}\Omega sin(\Omega t) + \frac{\frac{F_0}{m}(\omega_n^2-\Omega^2)}{(\omega_n^2-\Omega^2)^2+(2\zeta\omega_n\Omega)^2}\Omega cos(\Omega t) \\
0 &= -\zeta\omega_n(1)A(1) + (1)B(\sqrt{1-\zeta^2}\omega_n)(1) + \frac{\frac{F_0}{m}(\omega_n^2-\Omega^2)\Omega}{(\omega_n^2-\Omega^2)^2+(2\zeta\omega_n\Omega)^2}(1) \\
0 &= -\zeta\omega_n\frac{\frac{F_0}{m}(2\zeta\omega_n\Omega)}{(\omega_n^2-\Omega^2)^2+(2\zeta\omega_n\Omega)^2} + B(\sqrt{1-\zeta^2}\omega_n) + \frac{\frac{F_0}{m}(\omega_n^2-\Omega^2)\Omega}{(\omega_n^2-\Omega^2)^2+(2\zeta\omega_n\Omega)^2} \\
\mathbf{B} &= \frac{\frac{\mathbf{F_0}}{\mathbf{m}}\mathbf{\Omega}}{\sqrt{1-\zeta^2}\omega_{\mathbf{n}}}\left[\frac{(\mathbf{2}\zeta^{\mathbf{2}}\omega_{\mathbf{n}}^{\mathbf{2}})+\mathbf{\Omega^2}-\omega^{\mathbf{2}}}{(\omega_{\mathbf{n}}^{\mathbf{2}}-\mathbf{\Omega^2})^{\mathbf{2}}+(\mathbf{2}\zeta\omega_{\mathbf{n}}\mathbf{\Omega})^{\mathbf{2}}}\right]
\end{aligned}
$$

- substituting A & B back into the x(t) (3) to achieve the final equation for displacement of the system

(14)

$$
\begin{aligned}
\mathbf{x(t)} &= \mathbf{e}^{-\zeta\omega_{\mathbf{n}}\mathbf{t}}\left[\frac{\frac{\mathbf{F_0}}{\mathbf{m}}(\mathbf{2}\zeta\omega_{\mathbf{n}}\mathbf{\Omega})}{(\omega_{\mathbf{n}}^{\mathbf{2}}-\mathbf{\Omega^2})^{\mathbf{2}}+(\mathbf{2}\zeta\omega_{\mathbf{n}}\mathbf{\Omega})^{\mathbf{2}}}\right]\mathbf{cos}(\sqrt{1-\zeta^2}\omega_{\mathbf{n}}\mathbf{t}) + \\
&\mathbf{e}^{-\zeta\omega_{\mathbf{n}}\mathbf{t}}\frac{\frac{\mathbf{F_0}}{\mathbf{m}}\mathbf{\Omega}}{\sqrt{1-\zeta^2}\omega_{\mathbf{n}}}\left[\frac{\mathbf{2}\zeta^{\mathbf{2}}\omega_{\mathbf{n}}^{\mathbf{2}})+\mathbf{\Omega^2}-\omega_{\mathbf{n}}^{\mathbf{2}}}{(\omega_{\mathbf{n}}^{\mathbf{2}}-\mathbf{\Omega^2})^{\mathbf{2}}+(\mathbf{2}\zeta\omega_{\mathbf{n}}\mathbf{\Omega})^{\mathbf{2}}}\right]\mathbf{sin}(\sqrt{1-\zeta^2}\omega_{\mathbf{n}}\mathbf{t}) + \\
&\frac{\frac{\mathbf{F_0}}{\mathbf{m}}(-\mathbf{2}\zeta\omega_{\mathbf{n}}\mathbf{\Omega})}{(\omega_{\mathbf{n}}^{\mathbf{2}}-\mathbf{\Omega^2})^{\mathbf{2}}+(\mathbf{2}\zeta\omega_{\mathbf{n}}\mathbf{\Omega})^{\mathbf{2}}}\mathbf{cos(\Omega t)} + \frac{\frac{\mathbf{F_0}}{\mathbf{m}}(\omega_{\mathbf{n}}^{\mathbf{2}}-\mathbf{\Omega^2})}{(\omega_{\mathbf{n}}^{\mathbf{2}}-\mathbf{\Omega^2})^{\mathbf{2}}+(\mathbf{2}\zeta\omega_{\mathbf{n}}\mathbf{\Omega})^{\mathbf{2}}}\mathbf{sin(\Omega t)}
\end{aligned}
$$

Part C

MATLAB is used to solve the ODE (1) from Part A for the following constant values:

$$(\frac{F_0}{m}) = 1 \qquad \omega_n = 1 \qquad \zeta = 0.4 \qquad \Omega = [0.1, 0.5, 1, 2, 10]$$

Figure 1 demonstrates maximum displacement behaviour of the mass-damper system (from rest) at the resonance frequency. This occurs as a result of forcing function constructively interfering with the spring's natural frequency following the direction of movement for the full period. Damping controls the maximum displacement value and controls the constant max amplitude, since without damping the amplitude of oscillations (displacement) of the mass would increase with time. The forcing functions with frequencies close to the resonance frequency show large displacement values, i.e $\Omega = 0.5$. However the further away from natural frequency the lower the displacement values of the mass-damper system. These frequencies represent the destructive interference of the forcing function against the spring's movement.

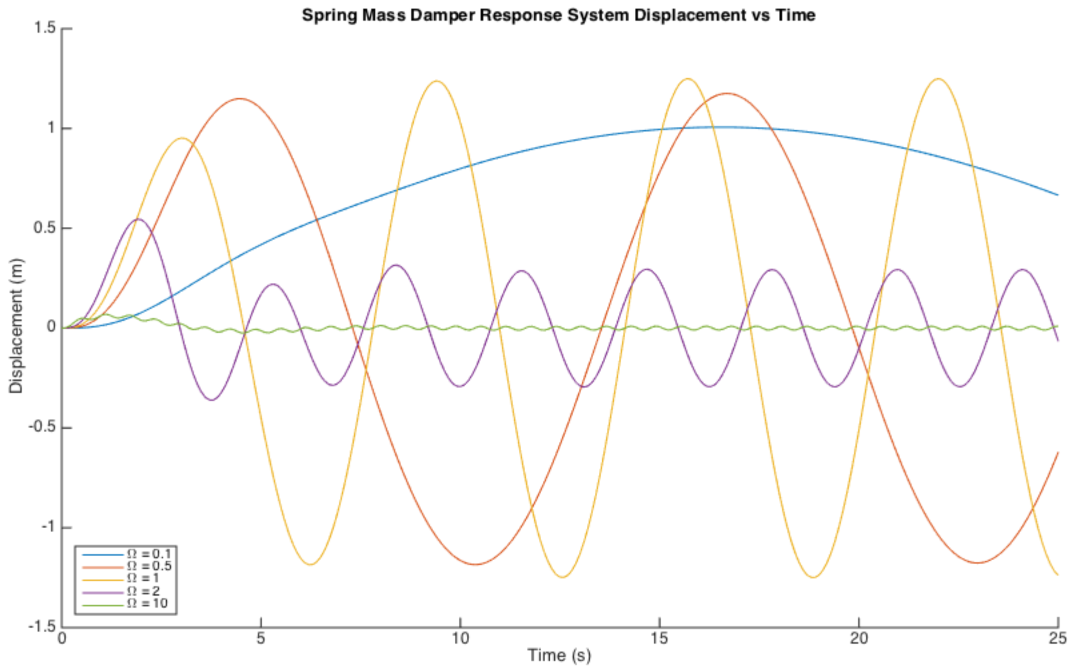

Figure 1: Overall response of the system for 5 values of forcing function frequency.

Part D

Consider a simple spring mass system equation with a periodic forcing function based on Dirac delta function.

$$x'' + \omega_n^2 x = \frac{F_0}{m}\sum_{n=1}^{\infty}\delta(t - nT) \qquad x(0) = x'(0) = 0 \tag{15}$$

- use Laplace Transform to solve for displacement

$$\begin{aligned}
\mathcal{L}\left\{x'' + \omega_n^2 x\right\} &= \mathcal{L}\left\{\frac{F_0}{m}\sum_{n=1}^{\infty}\delta(t - nT)\right\} \\
\mathcal{L}\left\{x''\right\} + \omega_n^2 X(s) &= \frac{F_0}{m}\sum_{n=1}^{\infty} e^{-nTs} \\
s^2X(s) - sx(0) - x'(0) + \omega_n^2 X(s) &= \frac{F_0}{m}\sum_{n=1}^{\infty} e^{-nTs} \\
X(s)(s^2 + \omega_n^2) &= \frac{F_0}{m}\sum_{n=1}^{\infty} e^{-nTs} \\
X(s) &= \frac{\frac{F_0}{m}\sum_{n=1}^{\infty} e^{-nTs}}{(s^2 + \omega_n^2)}
\end{aligned} \tag{16}$$

- take the inverse Laplace Transform

(17)

$$\mathcal{L}^{-1}\left\{X(s)\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{F_0}{m}\sum_{n=1}^{\infty} e^{-nTs}}{(s^2+\omega_n^2)}\right\}$$

$$x(t) = \frac{F_0}{m}\sum_{n=1}^{\infty}\mathcal{L}^{-1}\left\{\frac{e^{-nTs}}{\omega_n} \bullet \frac{\omega_n}{(s^2+\omega_n^2)}\right\}$$

$$\mathbf{x(t)} = \frac{\mathbf{F_0}}{\omega_\mathbf{n}\mathbf{m}}\sum_{\mathbf{n=1}}^{\infty}\left[\mathbf{sin}(\omega_\mathbf{n}(\mathbf{t}-\mathbf{nT}))\bullet\mathbf{H}(\mathbf{t}-\mathbf{nT})\right]$$

Part E

MATLAB is used to plot the function (17) from Part D for the following constant values:

$$(\tfrac{F_0}{m}) = 1 \qquad \omega_n = 1$$

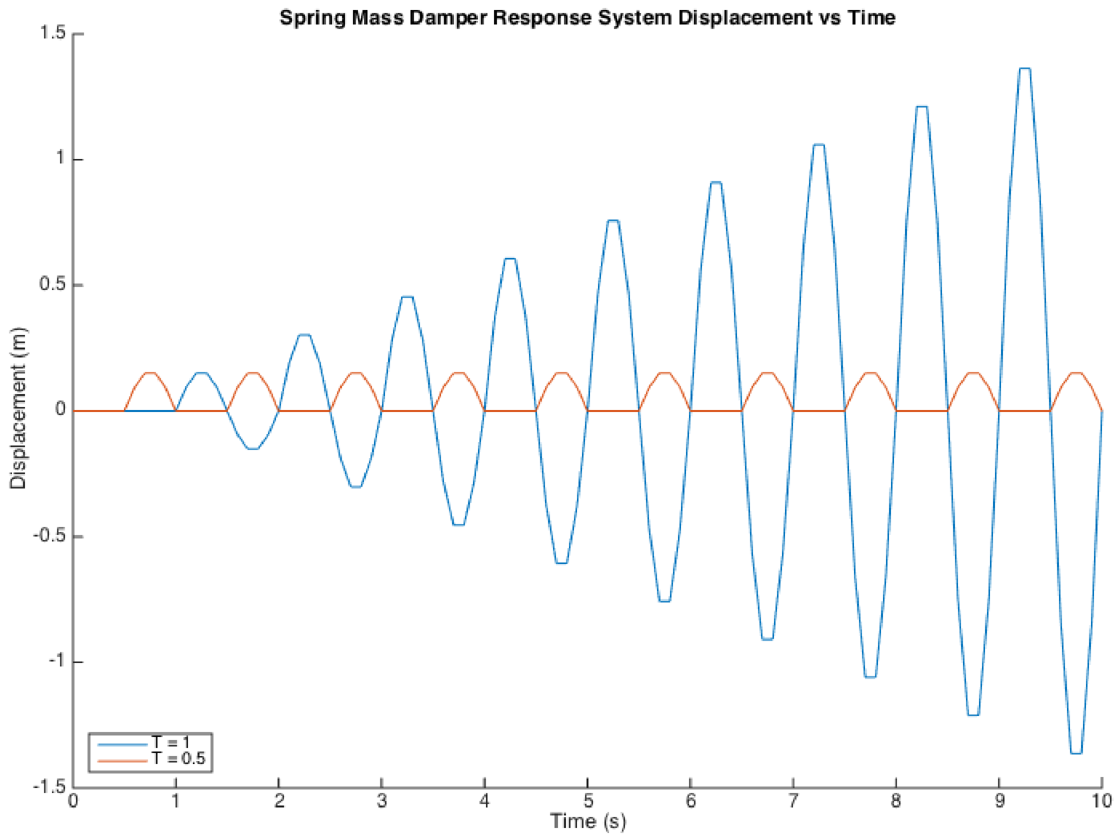

Figure 2: Overall response of the system when T = 1 such that the frequency of the pings = ω_n , and when T = 0.5 such that the frequency of pings = $2\omega_n$.

Figure 2 shows the constant increasing amplitude of displacement for (T = 1) the system as the period forcing function matches the natural frequency of the spring and constructively interferes with the mass's movement. Since the system does not contain damping, the amplitude of displacement continues to increase with each period of oscillation. However pings occur at twice the natural frequency, for period forcing function is consistently destructively interfering with the system movement as it sets a max allowable amplitude and returns the system to rest, not allowing displacement in the downward direction.

Appendix

Part C MATLAB Script

```
clear all;
% Denifition of Variables
F0 = 1;
m = 1;
Wn = 1;
Z = 0.4;
Omega = [0.1, 0.5, 1, 2, 10];

%solving the differential equation of mass-spring-damper system with
%different omega values
for i = 1:5
    syms x(t)
    Dx = diff(x);
    x(t) = dsolve(diff(x,2) + (2*Wn)*(Z*Dx) + (Wn^2)*x == (F0/m)*sin(Omega(i)*t),...
        x(0)==0, Dx(0)==0)
    %plotting spring-mass-damper system response
    figure(1)
    set(gca,'FontSize',12);
    plot(0:0.1:25,x(0:0.1:25));
    hold on
    clear x(t)
end

%labeling the graph
title('Spring Mass Damper Response System Displacement vs Time')
ylabel('Displacement (m)')
xlabel('Time (s)')
legend('\Omega = 0.1','\Omega = 0.5','\Omega = 1','\Omega = 2','\Omega = 10','Location','southwest')
box off
```

Part E MATLAB Script

```
clear all;
% Denifition of Variables
F0 = 1;
m = 1;
Wn = 2*pi;
T = [1, 0.5];

%plottting the Equation from Part D with diffrent periods
syms n;
for i = 1:2
    syms x(t)
    Dx = diff(x);
    x(t) = F0/(m*Wn) * symsum(sin(Wn*(t-n*T(i)))*heaviside(t-n*T(i)), n, 1, inf);
    %plotting spring-mass-damper system response
    figure(2)
    set(gca,'FontSize',12);
```

```
    plot(0:0.1:10,x(0:0.1:10));
    hold on
    clear x(t)
end

%labeling the graph
title('Spring Mass Damper Response System Displacement vs Time')
ylabel('Displacement (m)')
xlabel('Time (s)')
legend('T = 1','T = 0.5','Location','southwest')
box off
```