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PROJECT #1 FIRST ORDER ODE APPLICATIONS

This report involves modelling rocket's movement through manipulation of the **fundamental rocket equation** in two different scenarios. Scenario one involves only the effects of gravity on the rocket's movement, however scenario two also includes the drag force.

$$m\frac{dv}{dt} = -v_e\frac{dm}{dt} + F$$

PART A

- assuming only gravitational force act on the rocket

The force of gravity is $F = -mg$

The initial velocity of the rocket is $v(0) = 0$

Using the following notation:

m_r = mass of empty rocket

m_f = mass of fuel

$m_0 = m_r + m_f$ = total mass of the rocket and fuel at the time of launch,

- assuming fuel is consumed at constant rate of α, such that $m(t) = m_0 - \alpha t$ and v_e is treated as constant
- solving for equation of velocity using the fundamental rocket equation

$$m(t) = m_0 - \alpha t \text{, then } \frac{dm}{dt} = -\alpha$$

$$m(t)\frac{dv}{dt} = -v_e\frac{dm}{dt} + F, \text{ since } F = -mg \text{ and } \frac{dm}{dt} = -\alpha$$

$$m(t)\frac{dv}{dt} = -v_e(-\alpha) - m(t)g$$

$$\frac{dv}{dt} = \frac{v_e\alpha - m(t)g}{m(t)}$$

$$\frac{dv}{dt} = \frac{v_e\alpha}{m(t)} - g$$

$$dv = \left(\frac{v_e\alpha}{m_0 - \alpha t} - g\right)dt$$

$$\int dv = \int\left(\frac{v_e\alpha}{m_0 - \alpha t} - g\right)dt$$

$$v(t) = \int\frac{v_e\alpha}{m_0 - \alpha t}dt - \int g\,dt$$

$$v(t) = -v_e\int\frac{-\alpha}{m_0 - \alpha t}dt - gt + C$$

$$v(t) = -v_e\ln|m_0 - \alpha t| - gt + C$$

$$\text{since } v(0) = 0$$

$$0 = -v_e\ln|m_0 - \alpha(0)| - g(0) + C$$

$$0 = -v_e\ln|m_0| + C$$

$$C = v_e\ln|m_0|, \text{ assuming mass cannot be nevative absolute value signs are dropped}$$

$$\therefore v(t) = -v_e\ln|m_0 - \alpha t| - gt + v_e\ln(m_0)$$

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PART B

- assuming gravitational force and drag force act on the rocket
- using the same notation as Part A
- drag force is $F = -kv, \textit{where k is a constant}$
- therefore total force acting on the rocket is $F = -kv - mg$
- The initial velocity of the rocket is $v(0) = 0$

$$m(t)\frac{dv}{dt} = -v_e\frac{dm}{dt} + F, \quad \textit{since } F = -kv - mg \textit{ and } m(t) = m_0 - \alpha t, \quad \therefore \frac{dm}{dt} = -\alpha$$

$$m(t)\frac{dv}{dt} = -v_e(-\alpha) - kv - m(t)g$$

$$m(t)\frac{dv}{dt} + kv = -v_e(-\alpha) - m(t)g$$

$$\frac{dv}{dt} + \frac{kv}{m(t)} = \frac{-v_e(-\alpha)}{m(t)} - g$$

$$\frac{dv}{dt} + \frac{kv}{m_0 - \alpha t} = \frac{-v_e(-\alpha)}{m_0 - \alpha t} - g, \quad \textit{linear first order ODE, use the integrating factor method}$$

$$p(t) = \frac{k}{m_0 - \alpha t}$$

$$\sigma = e^{\int \frac{k}{m_0 - \alpha t} dt}$$

$$\sigma = e^{k\int \frac{1}{m_0 - \alpha t} dt}, \quad \textit{substitute } u = m_0 - \alpha t \quad \therefore \text{ du=-}\alpha\text{dt}$$

$$\sigma = e^{\frac{k}{-\alpha}\ln(m_0 - \alpha t)}$$

$$\sigma = (m_0 - \alpha t)^{-\frac{k}{\alpha}}$$

$$v(t) = \frac{1}{\sigma}\int \sigma q(t)\,dt + \frac{C}{\sigma}, \quad q(t) = \frac{v_e\alpha}{m_0 - \alpha t} - g$$

$$v(t) = \frac{1}{(m_0 - \alpha t)^{-\frac{k}{\alpha}}}\int (m_0 - \alpha t)^{-\frac{k}{\alpha}}\left[\frac{v_e\alpha}{m_0 - \alpha t} - g\right]dt + \frac{C}{(m_0 - \alpha t)^{-\frac{k}{\alpha}}}$$

$$v(t) = (m_0 - \alpha t)^{\frac{k}{\alpha}}\int \frac{v_e\alpha}{(m_0 - \alpha t)^{\frac{k}{\alpha}+1}} - \frac{g}{(m_0 - \alpha t)^{\frac{k}{\alpha}}}dt + C(m_0 - \alpha t)^{\frac{k}{\alpha}}$$

$$v(t) = (m_0 - \alpha t)^{\frac{k}{\alpha}}\int \frac{v_e\alpha}{(m_0 - \alpha t)^{\frac{k}{\alpha}+1}}dt - \int \frac{g}{(m_0 - \alpha t)^{\frac{k}{\alpha}}}dt + C(m_0 - \alpha t)^{\frac{k}{\alpha}} \qquad \textit{substitute } u = m_0 - \alpha t$$

$$\therefore \frac{du}{dt} = -\alpha, \; dt = \frac{du}{-\alpha}$$

$$v(t) = (m_0 - \alpha t)^{\frac{k}{\alpha}}\int \frac{v_e\alpha}{(u)^{\frac{k}{\alpha}+1}}\frac{du}{-\alpha} - \int \frac{g}{(u)^{\frac{k}{\alpha}}}\frac{du}{-\alpha} + C(m_0 - \alpha t)^{\frac{k}{\alpha}}$$

$$v(t) = (m_0 - \alpha t)^{\frac{k}{\alpha}}(-v_e)\int (u)^{-\frac{k}{\alpha}-1}du + \frac{g}{\alpha}\int (u)^{-\frac{k}{\alpha}}du + C(m_0 - \alpha t)^{\frac{k}{\alpha}}$$

$$v(t) = (m_0 - \alpha t)^{\frac{k}{\alpha}} \left[(-v_e) \left(\frac{(u)^{-\frac{k}{\alpha}}}{-\frac{k}{\alpha}} \right) + \frac{g}{\alpha} \left(\frac{(u)^{-\frac{k}{\alpha}+1}}{-\frac{k}{\alpha}+1} \right) \right] + C(m_0 - \alpha t)^{\frac{k}{\alpha}}$$

$$v(t) = (m_0 - \alpha t)^{\frac{k}{\alpha}} \left[\left(\frac{v_e}{\frac{k}{\alpha}(m_0 - \alpha t)^{\frac{k}{\alpha}}} \right) + \frac{g}{\alpha} \left(\frac{(m_0 - \alpha t)^{-\frac{k}{\alpha}+1}}{\frac{-k+\alpha}{\alpha}} \right) \right] + C(m_0 - \alpha t)^{\frac{k}{\alpha}}$$

$$v(t) = \frac{v_e \alpha}{k} + \frac{g}{\alpha - k}(m_0 - \alpha t) + C(m_0 - \alpha t)^{\frac{k}{\alpha}}$$

$$since \quad v(0) = 0$$

$$0 = \frac{v_e \alpha}{k} + \frac{g}{\alpha - k}(m_0 - \alpha(0)) + C(m_0 - \alpha(0))^{\frac{k}{\alpha}}$$

$$C(m_0)^{\frac{k}{\alpha}} = -\left[\frac{v_e \alpha}{k} + \frac{g m_0}{\alpha - k} \right]$$

$$C = \frac{-\left[\frac{v_e \alpha}{k} + \frac{g m_0}{\alpha - k} \right]}{(m_0)^{\frac{k}{\alpha}}}$$

$$v(t) = \frac{v_e \alpha}{k} + \frac{g}{\alpha - k}(m_0 - \alpha t) + \frac{-\left[\frac{v_e \alpha}{k} + \frac{g m_0}{\alpha - k} \right]}{(m_0)^{\frac{k}{\alpha}}}(m_0 - \alpha t)^{\frac{k}{\alpha}}$$

$$\therefore v(t) = \frac{v_e \alpha}{k} + \frac{g}{\alpha - k}(m_0 - \alpha t) - \left[\frac{v_e \alpha}{k} + \frac{g m_0}{\alpha - k} \right] \left(\frac{m_0 - \alpha t}{m_0} \right)^{\frac{k}{\alpha}}$$

PART C

This section of the report uses MATLAB software to model the ODE from Part A.

The following variables are set:

$$m_0 = 37.5kg$$

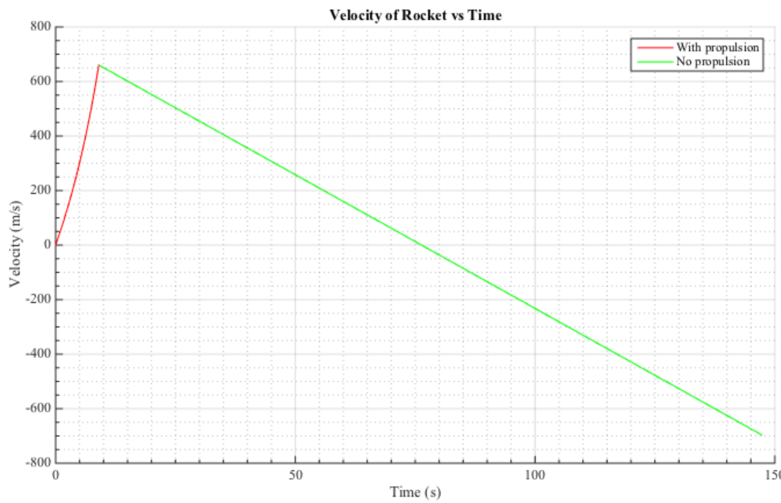
$$m_f = \frac{m_0}{2} kg$$

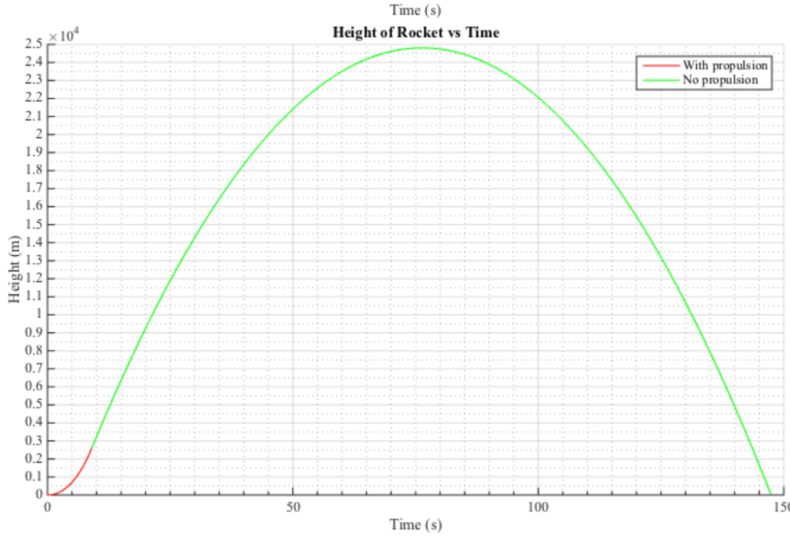
$$v_e = 1080 m/s$$

$$\alpha = \frac{m_f}{9 \sec}$$

- assume constant fuel consumption rate within 9 seconds
- assume vertical launch of the rocket from standing position on the earth's surface
- all forces but gravity are neglected

The following graphs model the rockets velocity and height of the rocket from t = 0s to t = t_r.





The red portion of the graphs models the rockets movement with propulsion, whoever the green portion describes the rockets motion when fuel is depleted and the rocket is traveling with no propulsion. From the velocity graph, the red portion shows non uniform acceleration (exponential) upwards, however the green portion is constant acceleration downwards due to gravity. On the position graph, excluding the red portion the rockets position shows symmetric parabolic shape, however the red portion is exponential increase in position while rocket is travelling with propulsion of the engine.

$\textit{Max Height} = 24{,}808 \text{ m}$ $\textit{Time at max height } t_{\max} = 76.31 \text{ s}$

$\textit{Terminal Velocity} = -697.67 \text{ m/s}$ $\textit{Time rocket returns to earth } t_r = 147.43 \text{ s}$

PART D

This section of the report uses MATLAB software to model the ODE from Part B.

The following variables are set:

$m_0 = 37.5kg$

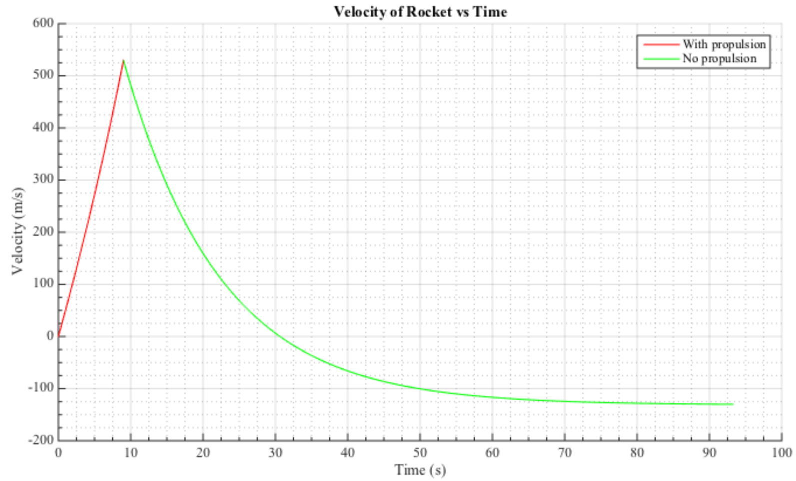
$m_f = \frac{m_0}{2} kg$

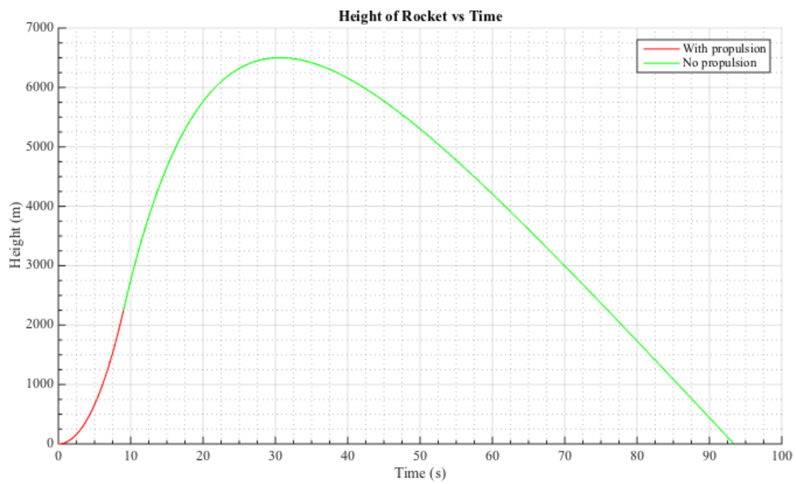
$v_e = 1080 m/s$

$\alpha = \frac{m_f}{9 \sec}$

- assume constant fuel consumption rate within 9 seconds
- assume vertical launch of the rocket from standing position on the earth's surface
- all forces but gravity and drag due to air resistance are neglected
- drag is proportional to rockets velocity with constant k = 1.4

The following graphs model the rockets velocity and height of the rocket from t = 0s to t = t_r.





The red portion of the graphs models the rockets movement with propulsion, whoever the green portion describes the rockets motion when fuel is depleted and the rocket is traveling with no propulsion. From the velocity graph, the red portion shows uniform acceleration upwards, however the green portion is non uniform acceleration downwards due to gravity and drag. On the position graph, the green is non uniform due to drag force opposing gravity when the rocket is falling back to earth.

$\textit{Max Height} = 6{,}500.5 \text{ m}$ $\quad$ $\textit{Time at max height } t_{max} = 30.64 \text{ s}$

$\textit{Terminal Velocity} = -130.17 \text{ m/s}$ $\quad$ $\textit{Time rocket returns to earth } t_r = 93.39 \text{ s}$

The results from part D are drastically different: the rocket's maximum height when drag forces due to air resistance are acting on it does not reach nearly as high as the rocket with only gravity force effect (24,808 m >> 6,500.5 m); the time of the entire flight of the rocket with drag force is decreased by approximately 63% compared to the rocket with no drag; the rocket in part D reaches terminal velocity of 103 m/s while the velocity of the rocket in part C increase until it hits the ground. The red portion in velocity graph of part D shows uniform acceleration of the rocket due to the fact that drag force is proportional to rockets velocity, however in part C the acceleration is non uniform and follows an exponential relationship. For the green portion of velocity graph the rocket with drag force experiences non uniform acceleration downwards, while the rocket with just gravity experiences constant acceleration. As for position, the rocket with drag force follows a non parabolic shape when the propulsion stops, while the rocket with no drag force creates a parabola.