Faculty of Engineering

ME 351 Analysis Project

Influence of Tube Length on Drainage System

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1 SIMULATION ANALYSIS

1.1 Methodology

This report outlines the analysis of water draining from a bottle through a discharge tube. The investigation focuses on the effect of varying the length of the discharge tube on the drainage time (i.e. the time for the water level to decrease from the initial height to the discharge tube height).

1.1.1 Assumptions

The simulation relies on the following assumptions:

1. The shape of the pop bottle is perfectly cylindrical for the middle portion, and the top portion of decreasing diameter is in the shape of an ellipsoid
2. The roughness of the straw (see simulation parameters)
3. The flow is considered stopped when the height reaches within 0.1% of its final value.
4. The flow begins as turbulent

1.1.2 MATLAB Script Procedure

The results were generated through a MATLAB script programmed to simulate the flow of water through a tube exiting a bottle. The script may be analyzed in two parts: initial velocity calculation, and the iterative process.

The simulation solves for the initial velocity of the water in the tube by assuming that the water starts as a turbulent flow. This is a reasonable estimate because the velocity is fastest at this point due to the reservoir pressure. This assumption means the friction factor (and Reynold's number) may be written as a function of V. After substitution, all the specific energies are represented as a function of V and solved using Bernoulli analysis using on F_{entry}, F_{tube}, and acc_{exit}. This may be represented as:

$$g(h(t) - H_3) = F_{entry} + F_{tube} + acc_{exit}$$
$$g(h(t) - H_3) = \frac{V^2(t)}{2}\left(\frac{K_1}{Re_d} + K_2\right) + \frac{V^2(t)}{2}\left(f\left(Re_d, \frac{\varepsilon}{d}\right)\frac{L}{d}\right) + \frac{V^2(t)}{2}$$

The initial condition *h(0) = 0.3*[m] is known, and this equation may be numerically solved for the initial velocity.

For the iterative process, the relationship between water level and flow exit velocity is given by the following function:

$$\frac{dh}{dt} = -\left(\frac{d}{D(h)}\right)^2 V(t)$$

Note that the bottle diameter is a non-constant function of height, this has been modelled by approximation and discussed in the following section. The function is solved using an iterative time-step approach, where the known height and Reynold's number at the previous interval, t_{i-1}, are used in Bernoulli's equation to numerically solve for the instantaneous velocity $V(t_i)$. This is found using MATLAB's `solve()` function to solve for V.

```
VSol = solve(F_e + F_t + V^2/2 - g*(h_previous - H3), V);
```

Next, the bottle diameter, a function of height, is determined using $h(t_{i-1})$. Both values are substituted into the differential equation and solved numerically using MATLAB's ode45() function.

```
[t_a, h_a] = ode45(@(t_a, v_a) -(d/D)^2 * eval(VSol), time - timestep:
timestep: time, h_previous);
```

Height is the last variable calculated per iteration, thus the time elapsed increases by the time-step value and the iterative process is repeated. The iteration will continue to repeat until the height is within a 0.1% tolerance of the final height, H_3=0.044m.

Simulation Parameters

The simulation is performed using the inputs summarized in Table 1.

Table 1: Simulation Parameters

Parameter	Symbol	Value	Notes
Bottle diameter	D	varies	See Diameter Function Derivation
Time step	t_{step}	0.1s	
Final height tolerance	tol	0.1%	Of steady-state height
Bottle height	h_0	0.300m	
Tube length	L	0.010m	
Tube diameter	d	0.007m	
Outlet height	H_3	0.044m	
Tube roughness	ε	0.0015 ± 60%	For a length of 1mm (Table 6.1 from Textbook [1])

1.1.3 Diameter Function Derivation

The diameter is a piecewise function with respect to the height (y). It may be represented as follows:

$$D(y) = \begin{cases} D_1(y) & \text{if } 0.3 > y > 0.15 \\ 0.109\text{m} & \text{if } y < 0.15 \end{cases}$$

It is the student's responsibility to estimate the variable diameter section, $D_1(y)$. To make a smooth transition between the two parts of the piecewise function an ellipse equation was used. Only the first quadrant of the ellipse was required, and thus, could be represented by the following function:

$$D_1 = \frac{b}{a}\sqrt{a^2 - y^2}$$

There are two known boundary conditions, which are substituted into the equation to solve for a and b.

$$D_1(0.3) = 0.022\text{m} \qquad \text{and} \qquad D_1(0.15) = 0.109\text{m}$$

Next, a horizontal and vertical shift was added to centre the ellipse at D=0.022m and y=0.15m. The final result is shown again in a piecewise function below. Figure 1 shows the shape of the pop bottle used in the simulation. The top curve of the pop bottle narrows the diameter of the bottle at 0.022m to 0.109m following a circular profile.

$$D(y) = \begin{cases} 0.58\sqrt{0.15^2 - (y - 0.15)^2} + 0.022 & \text{if } 0.3 > y > 0.15 \\ 0.109\text{m} & \text{if } y < 0.15 \end{cases}$$

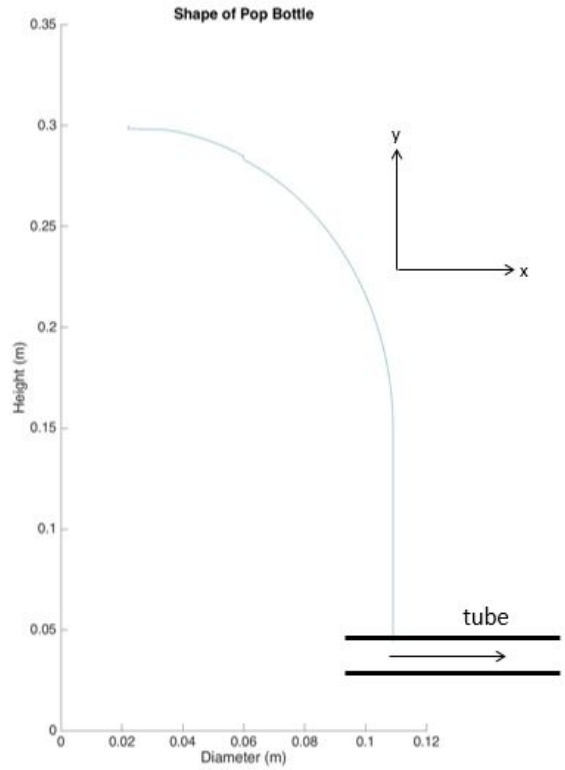

Figure 1: Pop Bottle Shape

1.1.4 Numerical Methods

Ode45 was used to solve the simulation's differential equations. Ode45 uses the numerical method Runge-Kutta, which requires a small-time step between solutions to guarantee a correct answer. The numerical methods used to solve the simulation do not affect the results because the time step is sufficiently small. Initially, the simulation showed a sudden drop in the height of the water in the first-time step, which was inconsistent with any physical explanation and the other results of the simulation (see Figure 2). When the time step was decreased, the numerical solution converged to the correct solution and the initial jump in the height disappeared, which confirmed that the time step caused the jump. Controlling the time step allowed us to minimize the effect of any error due to the implemented numerical method.

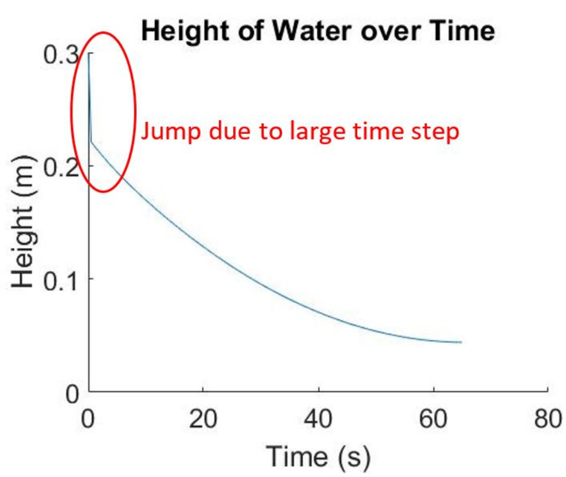

Figure 2: Initially sudden decrease in height due to small time step

1.2 Discussion

The results of the simulation are outlined and discussed in this section. All the simulation parameters are the same as in Table 1 unless otherwise stated.

1.2.1 To what extent is the flow in the drainage tube laminar during the draining time?

Figure 3 and Figure 4 show that the Reynold's number is directly proportional to the outlet velocity. This is consistent with the formula to calculate Reynold's number:

$$Re_d = \frac{\rho V d}{\mu}$$

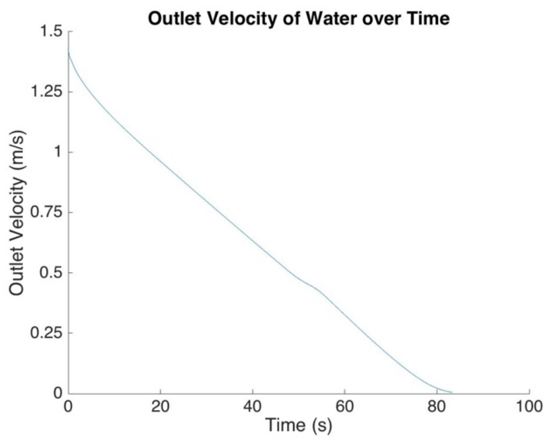

Figure 3: Output Velocity Throughout Simulation

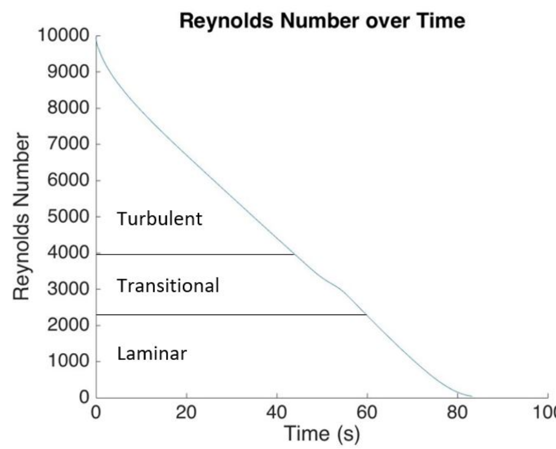

Figure 4: Reynold's Number Throughout Simulation

The flow is turbulent when the Reynold's number is greater than 4000, transitional when the Reynold's number is between 2300 and 4000, and laminar when the Reynold's number is less than 2300. These flow regions are shown in Figure 4. Thus, it is clear that flow is turbulent for the majority of the drain time. According to the simulation the Reynold's number reaches 2300 at 58.9 seconds. The corresponding velocity is 0.3296m/s. At this time the height is 0.0575m**.** Thus, the bottle has drained most of the water already. As the reservoir pressure and exit velocity decrease, the entry friction and tube friction decrease significantly. The reduced drag on the inner surface of the tube allows the flow to travel more linearly and smoothly. The ultimate result is laminar flow**.**

The time in which laminar flow is reached is dependent on the simulation parameters such as the tube diameter and the tube length. As the length of the tube is increased, the water drains slower and the slope of the Reynold's number as function of time decrease. This effect is demonstrated by changing the length of the tube and graphing the corresponding profile of the Reynold's number as shown in Figure 5. Ultimately, the flow in the drainage tube is only laminar for the last portion of the draining time.

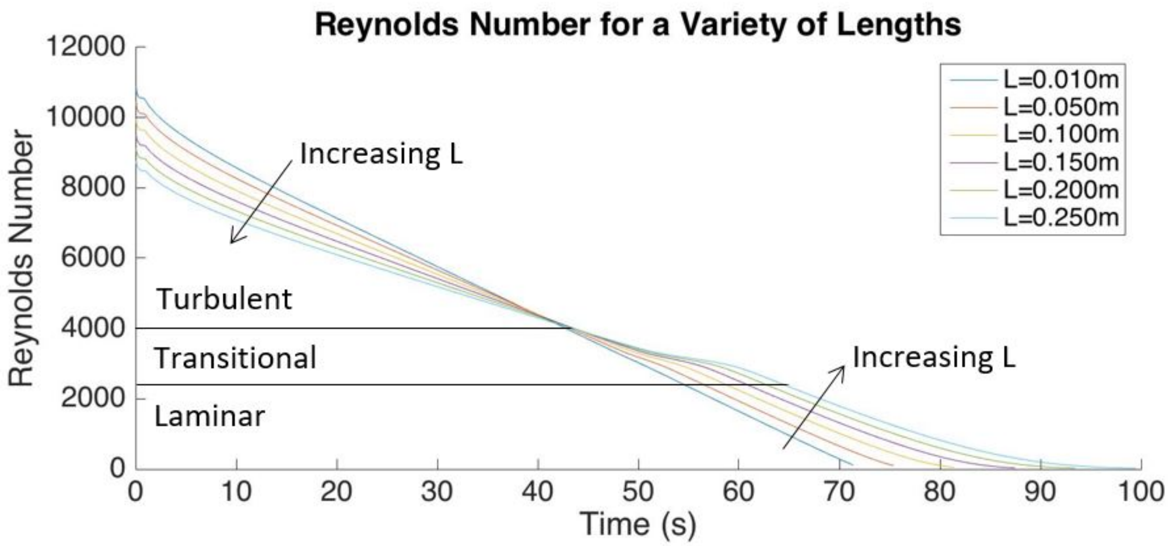

Figure 5: Reynold's Number for Various Lengths of Tube

1.2.2 To what extent is the influence of flow accelerations on pressure changes negligibly small?

The pressure change due to the flow acceleration is derived from the following formula:

$$\rho\vec{a} = -\nabla p$$
$$\rho\vec{a} = -\frac{\partial p}{\partial x} - \frac{\partial p}{\partial y}$$
$$\rho\vec{a} = -\frac{\partial p}{\partial x} - 0$$
$$\rho\vec{a} = -\frac{\partial p}{\partial x}$$
$$\frac{\partial p}{\partial x} = -\rho\vec{a}$$

Figure 6 shows the pressure change due to acceleration throughout the simulation. The acceleration of the fluid is in the order of 1m/s or less, which means the pressure gradient is in the order of -1000 Pa/m or less. This pressure change is not negligibly small. Consider the rate of change of pressure for a stagnant column of water. By substituting g for acceleration, the rate of change of pressure is:

$$-\rho g = -9810 \text{ Pa/m}.$$

Therefore, the acceleration at the beginning of the simulation influences the rate of change of pressure equivalent to approximately 10% of the effect of the height of the water in the pop bottle. If the effect of acceleration on the rate of change of pressure is negligible when the effect is less than 1% of that of gravity (i.e. when the acceleration is less than 0.0981 m/s^2), then for this simulation, acceleration is significant until 51.8 seconds and then negligible thereafter.

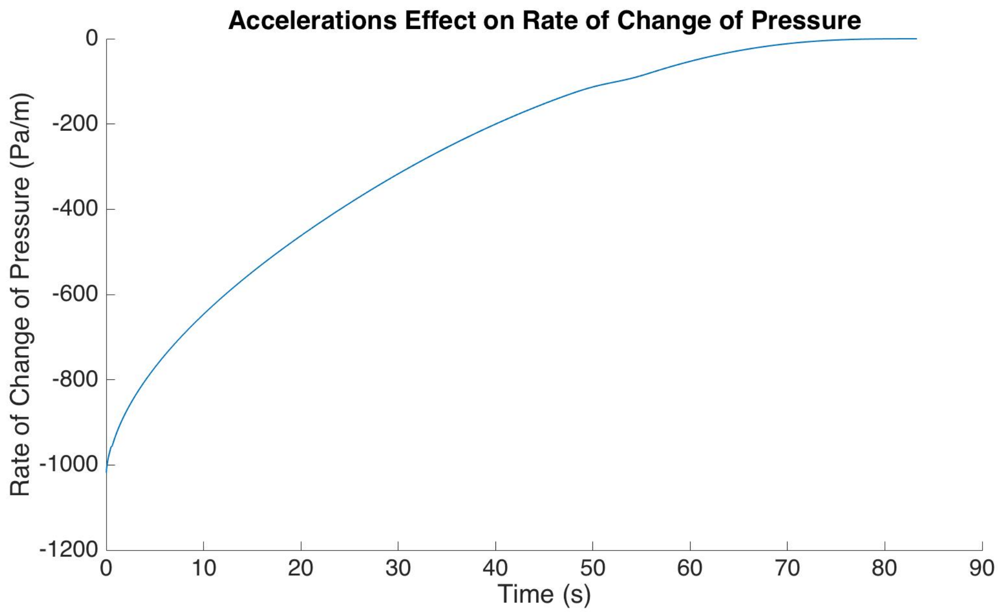

Figure 6: Graph of Pressure Changes in Tube in x-direction

1.2.3 To what extent is the influence of the friction in the flow entering the tube on pressure changes negligibly small?

Figure 7 shows the drainage time of the simulation with and without the effect of friction entering the tube. The bottle drains 21.5 seconds faster when the entry friction is neglected. This significant difference in drainage time shows that the effect of entry friction on drainage time is not negligibly small. Consequently, the total effect of entry friction is not negligible on the pressure change.

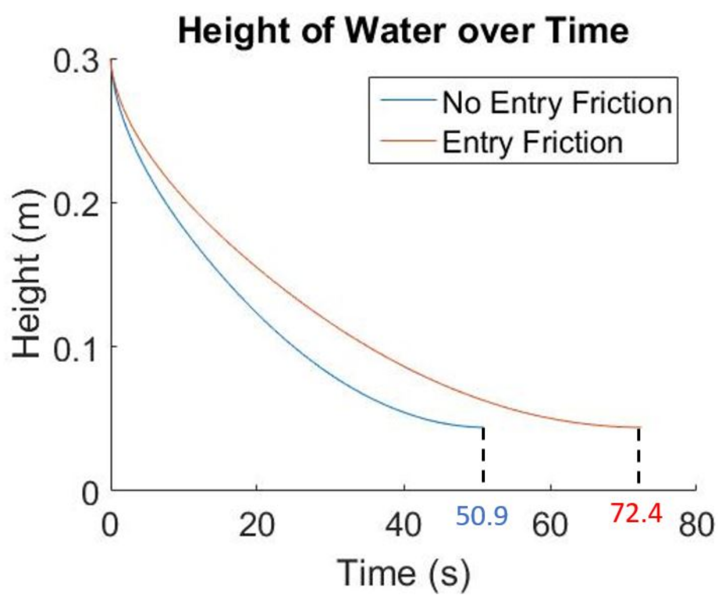

Figure 7: Drainage Times without F_entry

Figure 8 shows the specific energy of the entry friction, tube friction, an acceleration throughout the simulation. The tube is re-entrant which means that the entry friction is greater than if the tube was non-re-entrant because of the increased K-value. The entry friction term decays towards zero as the water level drops. This effect implies that the entry friction on pressure changes transitions from significant to negligible over the course of the experiment.

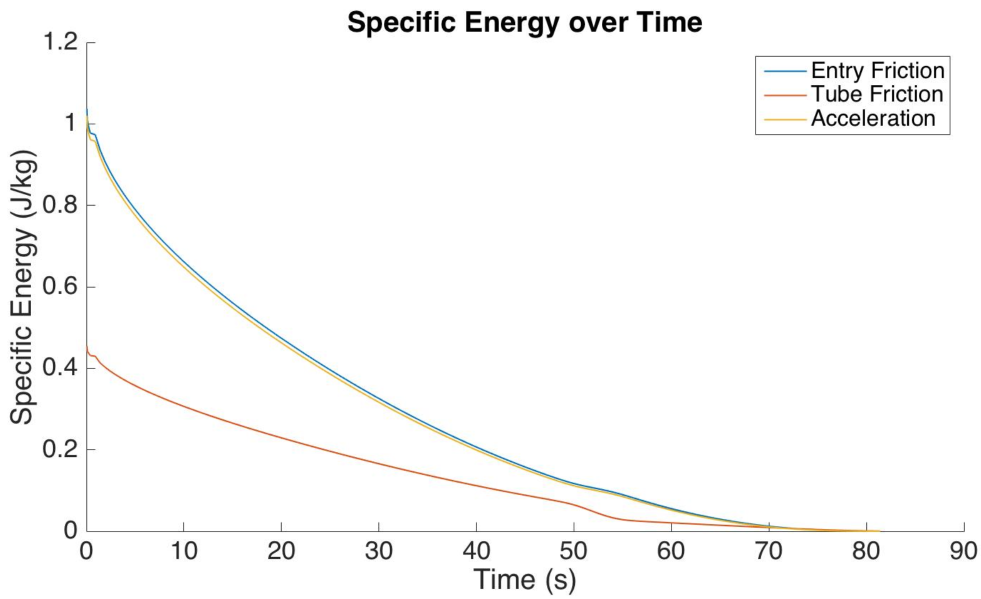

Figure 8: Specific Energy Terms Throughout Simulations

Therefore, the effect of entry friction on pressure change is significant at the beginning of the simulation and gradually decays until it is negligible when the water stops draining. Overall, the entry friction significantly affects the drainage time.

1.2.4 Most importantly from a design perspective, what is the relationship between the drainage time and the length of the discharge tube?

The length of the drainage tube does influence the drainage time. As the length of the drainage tube increases, the magnitude of the tube friction increases. Logically, as length increases more water is in contact with the inside surface of the tube, this results in the increased frictions. Figure 9 shows the relationship between the length of the tube and tube friction. Notice how increasing tube lengths result in increased friction.

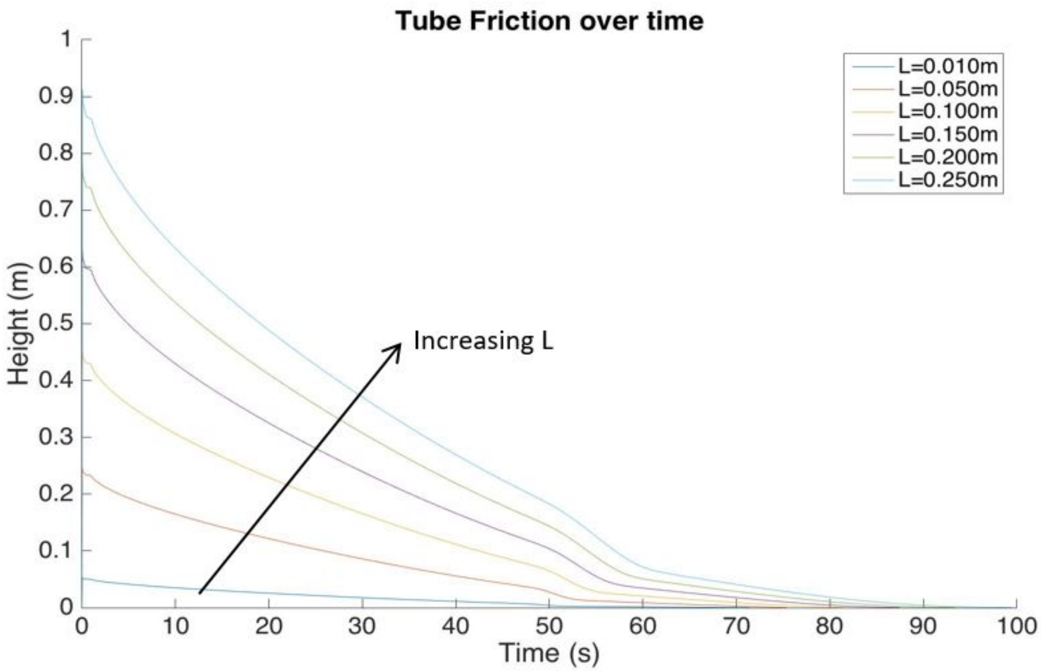

Figure 9: Tube Friction for Various Tube Lengths

The friction resists the velocity of the flow, so when the friction is decreased, it should allow the water in the tube to flow faster. This should result in a faster drainage rate. Figure 10 shows the relationship between the length of the tube and the drainage time. Clearly, the simulation models the theoretical response that increasing the length of the tube increases drainage time. Note that at a height of 0.44m, each drainage curve ends a different time.

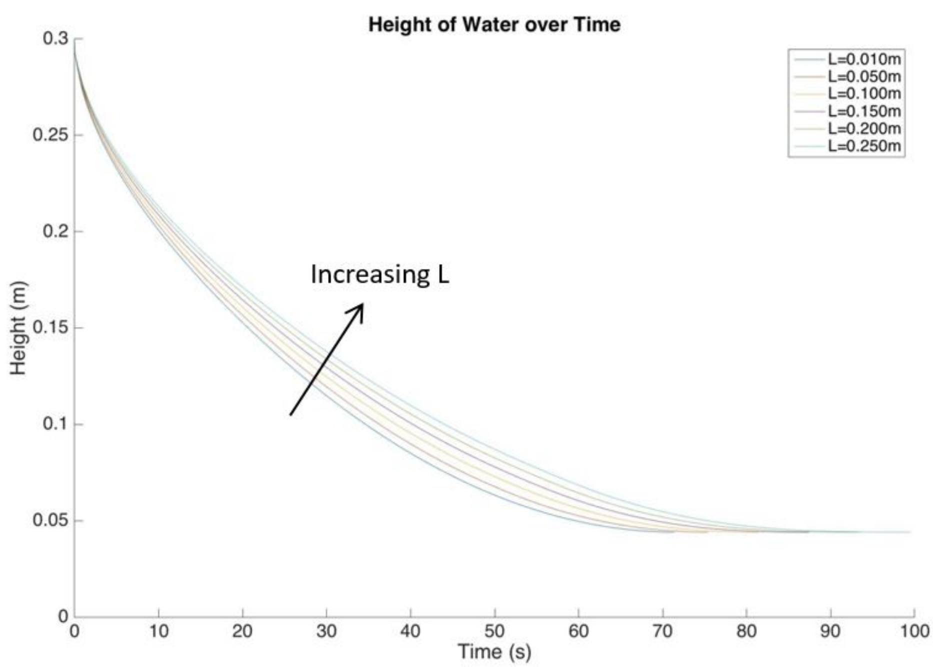

Figure 10: Drainage Time for Various Tube Lengths

The simulation data of total drainage time for each length is also provided as a scatter plot in Figure 11. A line of best fit between the data points have been added. There is clearly a linear relationship over the interval of 0.01m to 0.25m between the drain time and tube length.

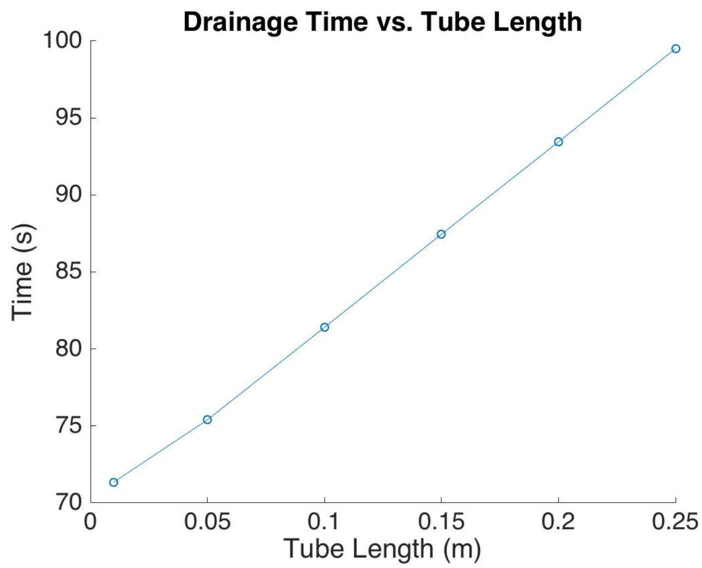

Figure 11: Total Elapsed Drain Time at Different Tube Lengths

Thus, it can be concluded that theoretical drain time is directly proportional to tube length for tubes of 7mm diameter.

1.3 Conclusions

This project uses Bernoulli analysis and an iterative differential equation solving approach to determine the water height of a pop bottle reservoir and the discharge flow rate through a straw tube.

It has been determined that the flow in the tube is mostly turbulent throughout the duration of drainage. Once the water level becomes low, the velocity decreases and the flow becomes laminar. This decrease in velocity is due to a negative x acceleration within the tube. As the water level approaches H3, the acceleration decreases. So, using the following equation:

$$\rho\overrightarrow{a_x} = -\frac{\partial \mathrm{p}}{\partial \mathrm{x}}$$

the rate of pressure change is proven to be negligible when the acceleration become negligible. The entry friction of a re-entrant tube has a significant effect on the drainage rate. Comparing it to the other portions, entry friction is much larger than tube friction, but is similar in magnitude to the acceleration. Finally, the tube length has been varied to prove that tube friction increases with the length. The consequence of increased friction is a slower drainage rate. The relationship between drain time and tube length is linear.

2 Enrichment Experiment Analysis

In the enrichment experiment, the reservoir and flow tube were modelled. Experimental results have been documented and will be analyzed with the simulation results to verify the correctness of the simulation.

2.1 Methodology

The set up involved cutting a hole in a 2L pop bottle and hot gluing a straw (d = 0.0056 m) to approximately the same height as in the video (4.4 cm) and as horizontal as possible. The water temperature was kept approximately constant at room temperature and the straw was horizontally supported during the drainage process. Figure 12 shows the experimental set up. Furthermore, the level of the water was kept in the constant diameter region of the water bottle to avoid errors in modelling of the diameter change with respect to height. The initial height of the water in the bottle was filled to 20 cm.

Figure 12: Experimental Selfie

Table 2 displays the drainage time for each tube length tested. The tube was shortened by increments of 2cm between each measurement. As the tube length decreases the drainage time decreases.

Table 2: Experimental Drainage Time Results

Tube Length (m)	Drain Time (s)
0.194	140.69
0.174	124.41
0.154	117.83
0.134	116.56
0.114	110.07
0.094	109.15

The experimental dimension parameters were inserted into the MATLAB script in attempt to precisely model the experiment conditions. Figure 13 compares the simulated and experimental results. The slope of the two curves are similar, this means that the simulated model precisely models the drainage but does have an offset error. There is also some experimental error in each measurement, as the line is not completely linear.

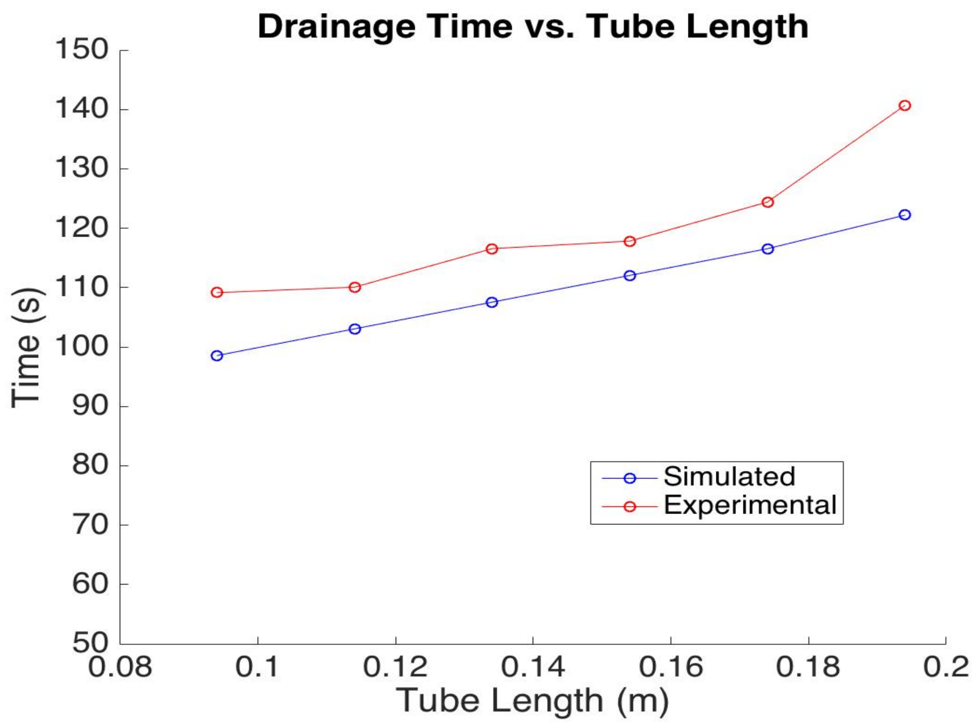

Figure 13: Comparison of Simulated and Experimental Results

The offset error between the two results could be explained by several factors:

- The error in the roughness of plastic (±60% according to Table 6.1 of the textbook [1])
- The error in measuring the dimensions of the experimental setup (mainly, the diameter of the tube and bottle)
- The difficulty of consistently measuring the time when the water stops flowing
- The error between the experimental height at which water stops flowing and the theoretical 0.1% of final height that the simulation uses to determine when the water stops flowing
- The error in the transient Reynold's number calculation, which can be as large as 2% (according to Equation 6.49 of the textbook [1])
- It cannot be concluded the water temperature was 20 degrees C because the experiment did not have access to a thermometer.

It can be concluded that the drainage time of the water increases with the length of the discharge tube. During the experiment the water never reached H3 due to surface tension and friction in the tube, thus, the time calculated is when the draining stopped.

3 REFERENCES

[1] F. M. White, Fluid Mechanics, Seventh Edition ed., M. Lange, Ed., New York, NY: McGraw-Hill, 2008, p. 862.

4 Appendix A: Single Simulation

```
clear all;
close all;
clc;

%% constants

% pop bottle sizing
D = 0.109;      % bottle diameter
h0 = 0.3;  % maximum height or h(t=0)

% tube sizing
d = 0.007;       % pipe diameter
L = 0.1; % between 0.01 and 0.25
H3 = 0.044; %height of bottom of the straw
eps = 0.0015/1000; %I believe this is the correct value for epsilon -Matt
g = 9.81;

% water properties
rho = 998;      % density of water
u = 0.001003;   % viscosity of water

% entry force properties
K_1 = 164;
K_2 = 1; %% since renentrant (0.5) for non re-entrant

%% symbolic variables
syms h(t);
syms V;

%% Setup Variables

maxSize = 1000;
t_ = zeros(maxSize,1);
h_ = zeros(maxSize,1);
V_ = zeros(maxSize,1);
Re_= zeros(maxSize,1);
D_ = zeros(maxSize,1);
flowType_ = -1*ones(maxSize,1);
F_t_=zeros(maxSize,1);
F_e_=zeros(maxSize,1);
V_a_=zeros(maxSize,1);

%% Initial Condition Calculations

f_l =   64/ Re;
f_tur = 1 / (-1.8 * log10(( eps/d /3.7)^1.11 + 6.9/Re))^2;
beta = 0.5 - 0.5*tanh((10*(Re - ((4000+2300)/2)))/(4000-2300));
f_tra = beta*f_l + (1-beta)*f_tur;

% Friction Entry
F_e = (V^2/2)*(K_1/Re + K_2);

%Friction Tube
F_t = V^2/2*L/d*f_tur; % assume turbulent
```

```
% solve for velocity based on height
V0 = solve(F_e + F_t + V^2/2 - g*(h0-H3), V);

Re_(1,1) = rho * V0 * d / u;

if( Re_(1,1) > 4000 )
    disp('The initial velocity yields a turbulent flow.')
end

%% Initial conditions
time = 0;
h_t = h0;
h_(1,1) = h0;
t_(1,1) = time;
V_(1,1) = V0;
D_(1,1) = 0.022;


%% Iterative Properties
timestep = 0.01;
index = 1;

a = (0.109-0.022)/2;
b = 0.15;
x0 = 0.022/2;
y0 = 0.15;


%% Iterative Process
while h_t >= H3 * (1 + 0.001)
    % Get previous values
    h_previous = h_(index,1);
    Re = Re_(index,1);

    %Increment time step and index
    time = time + timestep;
    index = index + 1;

    % Determine Friction Factor
    if Re < 2300
        f = 64 / Re;
        flowType_(index,1) = 0;
    elseif Re > 4000
        f = 1 / (-1.8 * log10(( eps/d /3.7)^1.11 + 6.9/Re))^2;
        flowType_(index,1) = 2;
    else
        f_l =   64/ Re;
        f_tur = 1 / (-1.8 * log10(( eps/d /3.7)^1.11 + 6.9/Re))^2;
        beta = 0.5 - 0.5*tanh((10*(Re - ((4000+2300)/2)))/(4000-2300));
        f = beta*f_l + (1-beta)*f_tur;
        flowType_(index,1) = 1;
    end

    % Friction Entry
    F_e = (V^2/2) * (K_1/Re + K_2);
```

```
    % Friction Tube
    F_t = V^2/2 * L/d * f;

    % Solve for velocity based on height
    VSol = solve(F_e + F_t + V^2/2 - g*(h_previous - H3), V);
    if length(VSol) > 1
        VSol = VSol(2,1);
    end

    % Determine the Diameter at this height
    if h_previous > 0.15
        D = 2.*(a./b.*sqrt(b.^2 - (h_previous - y0).^2) + x0);
    else
        D = 0.109;
    end

    %Numerically solve the ODE
    [t_arr, h_arr] = ode45(@(t_arr, h_arr) -(d/D)^2 * eval(VSol), time -
timestep: timestep: time, h_previous);
    %The final height of the ODE array is h(time)
    h_t = h_arr(end);

    %Populate arrays with data
    F_e_(index,1)= subs(F_e, V, VSol);
    F_t_(index,1)= subs(F_t, V, VSol);
    V_a_(index,1)= VSol^2/2;
    D_(index,1) = D;
    t_(index,1) = time;
    h_(index,1) = h_t;
    V_(index,1) = VSol;
    Re_(index,1)= rho * VSol* d / u;

    if (h_t < 0.285)
        %We can use a larger time step resoultion
        timestep = 0.1;
    end

    disp(['h(t=', num2str(time), ') = ', num2str(h_t) ]);

end

%% Plots
figure(1)
set(gca,'fontsize',18)
hold on
plot(t_(1:index,1), h_(1:index,1));
title('Height of Water over Time');
xlabel('Time (s)');
ylabel('Height (m)');
figure(2)
set(gca,'fontsize',18)
hold on
plot(t_(1:index,1), V_(1:index,1));
title('Outlet Velocity of Water over Time');
set(gca,'Ytick',0:0.25:1.5);
xlabel('Time (s)');
ylabel('Outlet Velocity (m/s)');
```

```
figure(3)
set(gca,'fontsize',18)
hold on
plot(t_(1:index,1), Re_(1:index,1));
title(['Reynolds Number over Time']);
xlabel('Time (s)');
ylabel(['Reynolds Number']);
figure(4)
set(gca,'fontsize',18)
hold on
plot(D_(1:index,1), h_(1:index,1));
title('Shape of Pop Bottle');
axis([0,0.12,0,0.35])
xlabel('Diameter (m)');
ylabel('Height (m)');
figure(5)
set(gca,'fontsize',18)
hold on
plot(t_(2:index,1), F_e_(2:index,1), 'LineWidth',1);
hold on
plot(t_(2:index,1), F_t_(2:index,1), 'LineWidth',1);
hold on
plot(t_(2:index,1), V_a_(2:index,1), 'LineWidth',1);
title('Specific Energy over Time');
xlabel('Time (s)');
ylabel('Specific Energy (J/kg)');
legend('Entry Friction','Tube Friction','Acceleration');
figure(6)
set(gca,'fontsize',18)
hold on
plot(t_(2:index,1), -1*rho.*V_a_(2:index,1), 'LineWidth',1);
title('Accelerations Effect on Rate of Change of Pressure');
xlabel('Time (s)');
ylabel('Rate of Change of Pressure (Pa/m)');
```