Centro UC
CLAPES UC
Centro Latinoamericano de Políticas Económicas y Sociales

On Estimating the Price of European Options: Time to Think Differently

Arturo Cifuentes
Bernardo Pagnoncelli

www.clapesuc.cl

Documento de Trabajo Nº 75

On Estimating the Price of European Options: Time to Think Differently

Arturo Cifuentes
Clapes UC, Santiago, CHILE
ac@arturocifuentes.com

Bernardo Pagnoncelli
Adolfo Ibáñez University, Santiago, CHILE
bernardo.pagnoncelli@uai.cl

JEL Codes: C02 C63 G13

Keywords: European options; Black and Scholes; calls; puts; derivatives

ABSTRACT

Estimating the price of European options is typically tackled based on the assumption that one needs to model the asset price evolution from t = 0 to T by means of a suitable stochastic process. Here we challenge that notion and demonstrate that one can derive expressions to estimate the prices of calls and puts bypassing completely the price-evolution process; in fact, one only needs to model the asset price distribution at T. The expressions we present here have been derived before by other authors using a different approach, namely, assuming that the asset price follows an arithmetic Brownian motion. Such derivation is of course correct, but it is much more mathematically involved. Our derivation only requires basic statistics. Anyhow, the key message is really an invitation to think about pricing European options in a different manner. That is, focusing on the price distribution at T, rather than the price evolution from 0 to T. In short, when dealing with European options what really matters is the destination, not the journey.

RESUMEN EJECUTIVO

La estimación del precio en las opciones europeas se trata normalmente partiendo del supuesto que es necesario modelar la evolución del precio del activo subyacente entre t = 0 y T por medio de un proceso estocástico adecuado. En este estudio desafiamos esa noción y demostramos que es posible derivar expresiones para estimar los precios de calls y puts ignorando por completo la evolución del precio; de hecho, solo es necesario modelar la distribución del precio del activo en T. Las expresiones que presentamos en este estudio han sido derivadas por otros investigadores anteriormente usando un método diferente, esto es, suponiendo que el precio del activo sigue un movimiento Bronwiano aritmético. Esa derivación es por supuesto correcta, pero es mucho más compleja desde un punto de vista matemático. Nuestra derivación solo se apoya en conceptos de estadística básica. De cualquier forma, en lo sustancial, nuestro estudio es una invitación a pensar en la estimación de los precios de las opciones europeas de una forma diferente. Esto es, focalizándose en la distribución del precio en T, más que en la evolución del precio de 0 a T. En síntesis, en lo que se refiere a opciones europeas, lo que importa es el destino final, y no la trayectoria.

THE PROBLEM

A European option is an option that can only be exercised at some future time $t = T$ (but not before). Typically, one knows: (1) the price of the asset today ($t = 0$), that is, X_0; (2) the risk-free rate (R); and (3) the strike (K). The problem consists of estimating the value of the corresponding options (call/put).

BACKGROUND

Currently, the Black-Scholes (B-S) equation, from which two valuation formulas are derived, is the standard reference to estimate the value of these options [1]. Derivation of the B-S equation is based on the assumption that the asset price evolves from 0 to T according to a stochastic process known as geometric Brownian motion (GBM); in short

$$dX = \lambda X\, dt + \omega X\, dz \qquad (1)$$

where X is the asset price; λ is its expected return; and dz represents a Wiener process, that is, $dz = \varepsilon \sqrt{dt}$ ($\varepsilon \sim N(0, 1)$). The term ω, known as volatility, is simply the standard deviation of the asset returns; a key assumption is that ω is constant. From Eq. (1) it follows that the asset returns are normally distributed. More interesting, invoking Ito's lemma and after some algebra it can be shown that the asset price is log-normally distributed, that is

$$\mathrm{Log}\,(X_T) \sim N(\mathrm{Log}[X_0] + \lambda - \omega^2 /2, \omega) \qquad (2)$$

Notwithstanding their merits, the B-S formulas are problematic to say the least, as they produce estimates that in many cases deviate significantly from market values. This situation is well known and it has been documented extensively [2-7]. The reason is the combination of two questionable assumptions: that the volatility is constant over the time interval (0, T), and that prices are log-normally distributed. Reality has not been kind to these two assumptions. In fact, a dirty secret of the financial markets is that the B-S formulas are not used to calculate the price of options. True, the formulas can be used to "estimate" the price, but that is it—the price is whatever the market dictates. And the market often disagrees with the B-S formulas. In fact, the most common use of the formulas is not to estimate prices, but to estimate implied volatilities. In this case the price of the option (given by the market) is taken as an input, and the formulas are solved for ω. Weatherall provides an interesting discussion of this point [8].

For our discussion, the key point to keep in mind is that the derivation on the B-S equation requires to model the evolution of the asset price from 0 to T, according to some stochastic process—a GBM in this case.

A DIFFERENT APPROACH

Since we are dealing with European options, that is, we only care about the price of the asset at T (i.e., X_T), we will make a radical assumption. We will only focus on estimating X_T, and completely ignore how X evolves from 0 to T. Furthermore, we assume that $X_T \sim N(\mu_T, \sigma_T)$. In other words, we are assuming that we have an estimate (μ_T) of the asset value at T, and we can think of σ_T as a parameter that describes the degree of accuracy of such estimate.

Call Option

Suppose that at T we have the right to buy this asset (call) for a price K. What should be a fair price, C, for this option, under a risk-neutral assumption? It would be the present value of the expected future cashflows discounted with the risk-free rate, R.

We note that the upside for this position, at T, is

$$C^* = \int_K^\infty (x - K)\phi(x)dx \qquad (3)$$

where $\phi(x)$ is the probability density function of a normal distribution with mean μ_T and standard deviation σ_T. Invoking (A.12), and defining $K^* = (K - \mu_T) / \sigma_T$, the above expression becomes

$$C^* = \int_K^\infty (x - K)\phi(x)dx \;\; = \;\; (\mu_T \; - K)\,(1 - N(K^*)) + \;\; \Delta(K^*) \qquad (4)$$

with $\Delta(\cdot)$ defined as in (A.7); $N(\cdot)$ represents the cumulative distribution function of the standard normal distribution.

Thus, a fair price for the call, C, at t = 0 is

$$C = \exp(-R)\,C^* \qquad (5)$$

Put Option

Suppose that at T we have the right to sell this asset (put) for a price K. In this case the downside for this position, at T, is

$$P^* = \; \int_{-\infty}^K (-K + x)\phi(x)dx \qquad (6)$$

which invoking (A.8) becomes

$$P^* = \; \int_{-\infty}^K (-K + x)\phi(x)dx \;\; = \;\; (\mu_T - K)\,N(K^*) - \;\; \Delta(K^*) \qquad (7)$$

Thus, a fair price for the put, P, at t = 0 is

$$P = \;\; -\exp(-R)\,P^* \qquad (8)$$

We know need to estimate μ_T and σ_T.

For μ_T (our estimate of X_T), we assume

$$\mu_T = X_0 \exp(R) \qquad (9)$$

which is consistent with a risk-neutral environment.

Regarding σ_T, we can think of it as the "uncertainty" of our estimate (μ_T).

Hence, let δ be the return of the asset between 0 and T. That is, $\delta = X_T/X_0 - 1$; it follows then that

$$\text{st. dev. }(\delta) = (1 / X_0)\ \{\text{st. dev. }(X_T)\} \tag{10}$$

which yields

$$\sigma_T = \text{st. dev. }(X_T) = X_0\ \{\text{st. dev. }(\delta)\} = X_0\ \omega \tag{11}$$

The value of ω, (the volatility in the B-S equation), can be estimated using past data. Therefore, we have that

$$X_T \sim N(X_0 \exp(R), X_0\ \omega) \tag{12}$$

In summary, we have arrived at two expressions to estimate the value of the call and put, Eqs. (5) and (8), ignoring completely how the asset price evolves from 0 to T and without invoking any arbitrage-based argument.

AN IMPORTANT OBSERVATION

It should be noted that some authors have investigated the possibility of using an arithmetic Brownian motion (ABM) instead of a GBM within the framework of the B-S equation [9-12]. That is,

$$dX = \alpha\ dt + \beta\ dz \tag{13}$$

where α and β are real numbers that represent the drift and volatility, respectively. Adopting an ABM implies (after some mathematical manipulation and invoking Girsanov's theorem), that X_T is normally distributed. That is

$$X_T \sim N(X_0 \exp(R), \Lambda) \tag{14}$$

and in this case, the standard deviation (Λ), is given by

$$\Lambda = \beta\sqrt{\frac{\exp(2R)-1}{2\,R}} \tag{15}$$

Clearly, the distributions expressed by Eqs. (12) and (14) are the same if $\Lambda = X_0\ \omega$, which means choosing β such that

$$\beta = \frac{X_0\omega}{\sqrt{\frac{\exp(2R)-1}{2\,R}}} \tag{16}$$

In other words, the pricing formulas given by Eqs. (5) and (8) could have been derived assuming that the asset priced follows an ABM in which β is given by Eq. (16). Such derivation, needless to say, is more mathematically involved [9-11].

In any event, it seems that the motivation behind the authors who derived pricing expressions analogous to Eqs. (5) and (8), but assuming an ABM instead of a GBM, was not really an attempt to challenge the B-S equation. It was really an attempt at capturing better the price evolution of real assets (as opposed to financial assets such as stocks). Alexander et al. provide an interesting discussion of this topic, and make the case that real assets can have negative prices, something that an ABM can accommodate but not a GBM [11]. It is perhaps due to this situation that no comparison has been made between option price estimates derived from a GBM assumption and an ABM assumption. The perception is that each distribution addresses a different issue, namely, the peculiar characteristics of different types of assets.

A PRACTICAL EXAMPLE

We contend that the expressions presented above to estimate the prices of calls and puts, in terms of accuracy, are neither better nor worse that the B-S estimates. Moreover, we claim that the difference between the estimates provided by the B-S formulas and Eqs. (5) and (8) are so insignificant, as to be immaterial when dealing with most realistic situations.

The following example illustrates these points. Consider a number of options on the S&P 500, with different strikes, as quoted on April 16, 2020 (data obtained from https://www.barchart.com/options). The expiration date is August 31, 2020. The risk-free rate for that period (based on the one-year US Treasury) was 0.00056; and the volatility (ω) estimated from data from the previous one-year period is 0.10080. The S&P 500 value (X_0) was 2,783.36. Table 1 is revealing.

First, it shows that the B-S estimates deviate significantly from market quotes for out-the-money options. This has been observed before, no surprise here. But second—and more important—it shows that the B-S values and those provided by Eqs. (5) and (8) are extremely similar, except for the deep out-of-the-money cases. However, in these cases, even though the discrepancies are huge (in relative terms), they are irrelevant as both estimates are off by a large margin. In short, from a practical standpoint, the estimates provided by the B-S formulas offer no advantage compared to those provided by Eqs. (5) and (8). It might be argued that the accuracy of the B-S predictions could have been improved had we relied on the market-implied volatility instead of historical volatility to estimate ω. However, had that been the case, the predictions resulting from Eqs. (5) and (8) would have benefited equally as well. Finally, the degree of agreement between the B-S estimates and those delivered by Eqs.(5) and (8) is not an artifact of the specific example we show; it persists when examining other cases (different stocks, different time-to-expiration windows, and different market periods).

Anyhow, this example is not an attempt at debunking the B-S formulas—the market has already done that to a large extent. The example simply intends to show that in most practical cases the B-S formulas do not perform better than the simpler formulas presented here.

CONCLUDING REMARKS

The B-S equation is still considered a major intellectual achievement within the financial engineering landscape. Its derivation is not trivial—it requires advanced stochastic calculus knowledge—and provides useful insights into the dynamics of options markets. However, it fails when it matters the most: it cannot provide reliable estimates of market prices. The humble formulas presented here—whose derivation only requires college-level statistics background—are worthy competitors in most real cases for they provide a comparable level of accuracy, but with much less fuss. That said, these formulas are unlikely to be useful in the context of high-frequency trading as they have been thought out with longer (at least a few days) periods in mind.

Additionally, the widely held notion that the log-normal distribution is better suited to capture stock prices behavior than the normal distribution should probably be reconsidered. If anything, both assumptions seem equally flawed (both result in poor approximations to market reality). But the argument that the log-normal distribution is better because it always gives non-negative prices—after considering the recent events in the oil futures market—has gone from being an advantage to being an inconvenience. Never mind that there is no evidence that prices fluctuate non-symmetrically around their expected values, as the log-normal distribution forces us to believe. Anyhow, a recent study by Brooks and Brooks regarding the advantages and disadvantages of using models based on the GBM and the ABM should give pause to anyone who believes in the GBM superiority [9].

However, there is a more profound and far more intriguing lesson that can be drawn from this exercise: <u>the formulas presented herein were derived completely ignoring (or bypassing) the way the asset price moves from $t = 0$ to T</u>. No attempt to model the underlying stochastic process was made. We only concentrated on estimating the value of the asset at T. This finding can, perhaps, point to a new approach to price European options: just aiming at modeling the price distribution at T and forget how "we get there." One obvious possibility is to explore the merits of different distribution functions (i.e., beyond the normal and log-normal) to describe the asset price behavior at T. Or to investigate different approaches to estimate ω (the volatility).

It is ironic, but maybe we have been barking at the wrong tree all along. That is, when modeling European options (as is the case when executing them) what really matters is the destination, not the journey. Let us focus on the destination in the future, and forget the journey.

CALLS						PUTS				
Market Quotes	Estimated by B-S	Estimated by Eq.(5)	Error (%) by B-S	Error (%) by Eq.(5)	X0 = 2783 STRIKE	Market Quotes	Estimated by B-S	Estimated by Eq.(8)	Error (%) by B-S	Error (%) by Eq.(8)
1765.0	1783.9	1783.9	-1.1	-1.1	**1000**	1.9	0.0	0.0	100.0	100.0
1664.5	1783.9	1783.9	-7.2	-7.2	**1100**	2.7	0.0	0.0	100.0	100.0
1567.2	1584.0	1584.0	-1.1	-1.1	**1200**	3.7	0.0	0.0	100.0	100.0
1469.0	1484.1	1484.1	-1.0	-1.0	**1300**	5.4	0.0	0.0	100.0	100.0
1371.3	1384.1	1384.1	-0.9	-0.9	**1400**	7.5	0.0	0.0	100.0	100.0
1274.0	1284.2	1284.2	-0.8	-0.8	**1500**	10.4	0.0	0.0	100.0	100.0
1176.0	1184.3	1184.3	-0.7	-0.7	**1600**	14.5	0.0	0.0	100.0	100.0
1078.8	1084.3	1084.3	-0.5	-0.5	**1700**	19.8	0.0	0.0	100.0	100.0
987.3	984.4	984.4	0.3	0.3	**1800**	26.8	0.0	0.0	100.0	99.9
898.2	884.4	884.5	1.5	1.5	**1900**	35.8	0.0	0.1	100.0	99.8
806.8	784.5	784.7	2.8	2.7	**2000**	46.9	0.0	0.2	99.9	99.5
720.6	684.7	685.2	5.0	4.9	**2100**	60.8	0.2	0.7	99.7	98.9
636.9	585.4	586.5	8.1	7.9	**2200**	76.1	0.8	1.9	98.9	97.5
558.0	487.5	489.4	12.6	12.3	**2300**	95.7	2.8	4.8	97.0	95.0
478.0	392.8	395.6	17.8	17.2	**2400**	117.5	8.1	10.9	93.1	90.7
403.2	304.1	307.4	24.6	23.7	**2500**	142.5	19.3	22.7	86.4	84.1
335.1	224.7	227.7	32.9	32.0	**2600**	169.2	39.9	42.9	76.4	74.6
266.6	157.8	159.4	40.8	40.2	**2700**	202.3	72.9	74.5	64.0	63.2
205.0	104.8	104.5	48.9	49.0	**2800**	242.8	119.9	119.6	50.6	50.7
152.7	65.8	63.6	56.9	58.3	**2900**	285.8	180.8	178.7	36.7	37.5
104.7	39.0	35.7	62.8	65.8	**3000**	340.6	253.9	250.7	25.4	26.4
65.9	21.8	18.4	66.9	72.1	**3100**	403.0	336.7	333.3	16.4	17.3
37.5	11.6	8.6	69.1	77.0	**3200**	475.2	426.4	423.5	10.3	10.9
20.4	5.8	3.7	71.5	82.0	**3300**	557.6	520.6	518.4	6.6	7.0
10.6	2.8	1.4	73.8	86.8	**3400**	646.5	617.5	616.1	4.5	4.7
5.6	1.3	0.5	77.2	91.3	**3500**	740.3	715.9	715.2	3.3	3.4
3.2	0.6	0.1	82.6	95.3	**3600**	834.5	815.2	814.8	2.3	2.4
2.0	0.2	0.0	88.3	97.9	**3700**	937.7	914.8	914.6	2.4	2.5
1.4	0.1	0.0	93.4	99.3	**3800**	1036.9	1014.6	1014.5	2.1	2.2
1.0	0.0	0.0	96.5	99.8	**3900**	1137.6	1114.5	1114.5	2.0	2.0
0.7	0.0	0.0	98.1	99.9	**4000**	1237.1	1214.4	1214.4	1.8	1.8
0.6	0.0	0.0	101.4	100.2	**4100**	1336.7	1314.3	1314.3	1.7	1.7
0.5	0.0	0.0	107.0	100.7	**4200**	1430.3	1414.3	1414.3	1.1	1.1

TABLE 1: Options on the S&P 500 with an expiration date of August 31, 2020. The table shows: (i) the market quotes as of April 16, 2020; (ii) the estimates provided by the B-S formulas; and (iii) the estimates provided by Eqs.(5) and (8). The value of the S&P 500 index (X_0) at t = 0 (April 16) is 2783.

REFERENCES

[1] Black, F. and Scholes, M. (1973), The pricing of options and corporate liabilities, Journal of Political Economy, 81(3), 637-654.

[2] Srivastava, A. and Shastri, M. (2018), A study of relevance of Black-Scholes model on option prices of Indian stock market, International Journal of Governance and Financial Intermediation, 1(1), 82–104.

[3] Huhta, T. (2017), Performance of the Black-Scholes option pricing model, empirical evidence on S&P 500 call options in 2014, University of Oulu, Business School.

[4] Gustafsson M. and Morck, E. (2010), Black-Scholes option pricing formula, School of Business, Economics and Law, Goteborg University.

[5] Gencay, R. and Salih, A.. (2003), Degree of mispricing with the Black-Scholes model and nonparametric cures, Annals of Economics and Finance, 4,73-101.

[6] Yakoob, M. (2002), An empirical analysis of options valuation techniques using stock index options, Duke University.

[7] Fortune, P. (1996), Anomalies in option pricing: the Black-Scholes model revisited, New England Economic Review, March/April, 17-40.

[8] Weatherall, J. O. (2017), The Peculiar Logic of the Black-Scholes Model, Philosophy of Science, 85(5), December.

[9] Brooks, R. and Brooks, J. (2016), Option valuation based on arithmetic Brownian motion: justification and implementation issues, SSRN website, paper # 2228288.

[10] Kolman, M. (2013), Black-Scholes model under arithmetic Brownian motion, University of Economics, Prague, SSRN website, paper # 2332640.

[11] Alexander, D. R. et al. (2012), Arithmetic Brownian motion and real options, European Journal of Operations Research, 219, 114-122.

[12] Liu, Q. (2007), Options prices under arithmetic Brownian motion and their implications for modern derivatives pricing, SSRN website, paper # 959809.

APPENDIX

Let ϕ(x) be the probability density function of a normal distribution with parameters μ (mean) and σ (standard deviation).

We seek to find an expression to calculate

$$\int_{-\infty}^{K} (-K + x)\phi(x)dx \tag{A.1}$$

If we define K* = (K – μ) / σ, then, the first term in the integral becomes

$$\int_{-\infty}^{K} (-K\ )\phi(x)dx = -K\ N(\ K^*) \tag{A.2}$$

in which N(·) represents the cumulative distribution function of the standard normal distribution.

Performing the following change of variable, Y = (X – μ) / σ, the second term in the integral (A.1) can be rewritten as

$$\int_{-\infty}^{K} x\ \phi(x)dx = \int_{-\infty}^{K^*} (\ \sigma y + \ \mu\ )\ \eta(y)dy \tag{A.3}$$

where *η(y)* is the density function of the standard normal distribution, that is

$$\eta(y) = \frac{1}{\sqrt{2\,\pi}}\ exp(-\left(\frac{1}{2}\right)\ y^2\ ) \tag{A.4}$$

And by noting that

$$\int y\eta(y)dy = -\eta(y) \tag{A.5}$$

we have that

$$\int_{-\infty}^{K} x\ \phi(x)dx =$$

$$\mu\, N(K^*) - \frac{\sigma}{\sqrt{2\,\pi}}\; exp(-\left(\frac{1}{2}\right)(K^*)^2) = \;\mu\, N(K^*) - \Delta(K^*) \qquad \text{(A.6)}$$

where

$$\Delta(u) \;= \frac{\sigma}{\sqrt{2\,\pi}}\; exp(-\left(\frac{1}{2}\right)u^2\;) \;=\; \sigma\,\eta(u) \qquad \text{(A.7)}$$

Finally, invoking (A.2) and (A.6), the integral in (A.1) can be conveniently evaluated by means of the following expression

$$\int_{-\infty}^{K} (-K + x)\phi(x)dx \;=\; (\mu - K)\, N(K^*) - \;\Delta(K^*) \qquad \text{(A.8)}$$

We now need an expression to evaluate

$$\int_{K}^{\infty} (x - K)\,\phi\,(x)dx \qquad \text{(A.9)}$$

The first term in the integral, performing the same change of variable as before and with a similar manipulation, can be expressed as

$$\int_{K}^{\infty} x\,\phi(x)dx \;=\; \mu\,(1 - N(K^*)) \;+\; \Delta(K^*) \qquad \text{(A.10)}$$

And noting that

$$\int_{K}^{\infty} (-K\;)\phi(x)dx = \;-K\,(1 - N(\,K^*)\,) \qquad \text{(A.11)}$$

using the results from (A.10) and (A.11), we get

$$\int_{K}^{\infty} (x - K)\,\phi\,(x)dx \;=\; (\mu\; - K)\,(1 - N(K^*)) + \;\Delta(K^*) \qquad \text{(A.12)}$$

Centro UC
CLAPES UC
Centro Latinoamericano de Políticas Económicas y Sociales

clapesuc

@clapesuc

claples_uc

clapesuc